Distributed Nash Equilibrium Seeking over Time-Varying Directed Communication Networks

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Duong Thuy Anh Nguyen
Arizona State University

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Duong Thuy Anh Nguyen, *Student Member, IEEE*, Duong Tung Nguyen, *Member, IEEE*, and Angelia Nedić, *Member, IEEE*

***Abstract*—We study distributed algorithms for finding a Nash equilibrium (NE) in a class of non-cooperative convex games under partial information. Specifically, each agent has access only to its own smooth local cost function and can receive information from its neighbors in a time-varying directed communication network. To this end, we propose a distributed gradient play algorithm to compute a NE by utilizing local information exchange among the players. In this algorithm, every agent performs a gradient step to minimize its own cost function while sharing and retrieving information locally among its neighbors. The existing methods impose strong assumptions such as balancedness of the mixing matrices and global knowledge of the network communication structure, including Perron-Frobenius eigenvector of the adjacency matrix and other graph connectivity constants. In contrast, our approach relies only on a reasonable and widely-used assumption of row-stochasticity of the mixing matrices. We analyze the algorithm for time-varying directed graphs and prove its convergence to the NE, when the agents' cost functions are strongly convex and have Lipschitz continuous gradients. Numerical simulations are performed for a Nash-Cournot game to illustrate the efficacy of the proposed algorithm.**

***Index Terms*—Nash equilibrium, game theory, time-varying directed graphs, distributed algorithms.**

I. Introduction

Game theory provides a framework to understand decision making in strategic situations where multiple agents aim to optimize their individual, yet interdependent, objective functions. The notion of Nash equilibrium (NE) in non-cooperative games characterizes desirable and stable solutions to the games, which subsequently can be used to predict the agents' individual strategies and payoffs. A NE is a joint action from which no agent has an incentive to unilaterally deviate. Indeed, non-cooperative games have been extensively studied to address various engineering problems in different areas, such as communication networks, electricity markets, power systems, flow control, and crowdsourcing [1], [11], [16], [20], [26]. Hence, developing efficient NE seeking algorithms has drawn increasing attention in recent years. In this paper, based on the distributed gradient play approach, we develop a discrete-time algorithm to find a NE in a non-cooperative game played over time-varying directed communication networks.

In classical non-cooperative complete information game theory, the payoff of each agent is determined by its own actions and the observations of the other agents' actions. Thus, a large body of existing work, using best-response or gradient-based schemes, requires each agent to know the competitors' actions to search for a NE [2], [28], [29]. However, this full-decision information assumption is impractical in many engineering systems [17], for example, the Nash-Cournot competition [7]. Recently, there has been extensive research conducted on fully distributed algorithms, which rely on local information only (i.e., the partial-decision information setting [3]), to compute NE. However, most of the proposed algorithms are built upon the (projected) gradient and consensus dynamics approaches, in both continuous time [9], [27] and discrete time [12], [22]. Also, they are based on the information available to the agents and need certain properties of the agents' cost functions, such as convexity, strong monotonicity, and Lipschitz continuity.

In [17], the authors propose a gradient-based gossip algorithm for distributed NE seeking in general non-cooperative games. For a diminishing stepsize, this algorithm converges almost surely to a NE under strict convexity, Lipschitz continuity, and bounded gradient assumptions. With the further assumption of strong convexity, a constant stepsize α guarantees the convergence to an $O(\alpha)$ neighborhood of the NE. In [19] an algorithm within the framework of the inexact-ADMM is developed and its convergence rate $o(1/k)$ is established for a fixed stepsize under the co-coercivity assumption on the game mapping. Reference [23] provides an accelerated version of the gradient play algorithm (Acc-GRANE) for solving variational inequalities. The analysis is based on strong monotonicity of a so-called augmented mapping which takes into account both the gradients of the cost functions and the communication settings. However, this algorithm is applicable only to a subclass of games characterized by a restrictive connection between the agents, Lipschitz continuity, and strong monotonicity constants. By assuming the restricted strong monotonicity of the augmented game mapping, in [24], the authors show that this algorithm can be applied to a broader class of games and demonstrate its geometric convergence to a NE. However, both types of the procedures mentioned above require a careful selection of both stepsize and the augmented mapping. Alternatively, by leveraging contractivity properties of doubly stochastic matrices, in [22], the authors develop a distributed gradient-play based scheme whose convergence properties do not depend on the augmented mapping. Nevertheless, all the methods cited above are designed for time-invariant undirected networks.

There is a growing interest in studying NE seeking for communication networks with switching topologies. The early works [12], [13] consider aggregative games over time-varying, jointly connected, and undirected graphs. This result is extended in [4] to games with coupling constraints. In [18], an asynchronous gossip algorithm to find a NE over a directed

The authors are with the School of Electrical, Computer and Energy Engineering, Arizona State University, Tempe, AZ, United States. Email: {dtnguy52, duongnt, Angelia.Nedich}@asu.edu. *Corresponding Author*: Angelia Nedić.

graph is developed under the assumption that every agent is able to update all the estimates of the agents who interfere with its cost function. In [6], a projected pseudo-gradient based algorithm is proposed that works for time-varying, weight-balanced, and directed graphs. The balancedness assumption is relaxed in follow-up work [5], where a modified algorithm is proposed which requires global knowledge of the communication graph structure, including the Perron-Frobenius eigenvector of the adjacency matrix and a constant related to the graph connectivity. However, constructing weight balanced matrices even in a directed static graph is non-trivial and computationally expensive [10], making it impractical for time-varying directed graphs. Also, the knowledge of the global communication structure is a demanding assumption since the computation of the Perron-Frobenius eigenvector in every iteration imposes significant computational burden.

Contributions. Motivated by the penetration of the game theoretic approaches into cooperative control and distributed optimization problems in engineering systems where full communication is not available [3], [6], [22], [23], this paper addresses NE seeking under the so-called partial-decision information scenario. Agents only have access to their own cost functions and local action sets, and engage in nonstrategic information exchange with their neighbors in a network. Our contributions are summarized as follows:

- We propose a fully-distributed algorithm to compute a NE over time-varying directed communication networks. While previous works assumed balancedness, or the knowledge of some global communication network parameters, our approach only requires the usual row-stochasticity assumption on the weights. The algorithm is simple to implement in a distributed fashion as each agent can locally decide on the weights for the information received from its neighbors.
- The proposed algorithm does not depend on any parameter related to the network structure, such as the Perron-Frobenius eigenvector of the adjacency matrix or any other constant related to the graph connectivity. Moreover, the convergence analysis of our approach does not rely on the augmented mapping used in [22], [24]; instead, the convergence of the method is focused on the choice of the stepsize values.
- We prove that the algorithm is guaranteed to converge to a NE under mild assumptions of convexity, strong monotonicity, and Lipschitz continuity of the game mapping.

II. Notations and Terminologies

Throughout this paper, all vectors are viewed as column vectors unless stated otherwise. We consider real normed space E, which is either space of real vectors $E = \mathbb{R}^n$ or the space of real matrices $E = \mathbb{R}^{n\times n}$. For every vector $u \in \mathbb{R}^n$, u' is the transpose of u. We use $\langle \cdot, \cdot\rangle$ to denote the inner product, and $\|\cdot\|$ to denote the standard Euclidean norm. We write $\mathbf{0}$ and $\mathbf{1}$ to denote the vector with all entries equal to 0 and 1, respectively. The dimensions of the vectors $\mathbf{0}$ and $\mathbf{1}$ are to be understood from the context.

In this paper, we consider a discrete time model where the time index is denoted by k. The i-th entry of a vector u is denoted by u_i, while it is denoted by $[u_k]_i$ for a time-varying vector u_k. Given a vector v, we use $\min(v)$ and $\max(v)$ to denote the smallest and the largest entry of v, respectively, i.e., $\min(v) = \min_i v_i$ and $\max(v) = \max_i v_i$. We write $v > \mathbf{0}$ to indicate that the vector v has positive entries. A vector is said to be a stochastic vector if its entries are nonnegative and sum to 1. For a set S with finitely many elements, we use $|S|$ to denote its cardinality.

To denote the ij-th entry of a matrix A, we write A_{ij}, and we write $[A_k]_{ij}$ when the matrix is time-dependent. For any two matrices A and B of the same dimension, we write $A \le B$ to denote that $A_{ij} \le B_{ij}$, for all i and j; in other words, the inequality $A \le B$ is to be interpreted component-wise. A matrix is said to be nonnegative if all its entries are nonnegative. For a nonnegative matrix A, we use $\min(A^+)$ to denote the smallest positive entry of A, i.e., $\min(A^+) = \min_{\{ij:A_{ij}>0\}} A_{ij}$. A nonnegative matrix is said to be row-stochastic if each row entries sum to 1, and it is said to be column-stochastic if each column entries sum to 1. In particular, if $A \in \mathbb{R}^{n\times n}$ is row-stochastic and $B \in \mathbb{R}^{n\times n}$ is column-stochastic, then $A\mathbf{1} = \mathbf{1}$ and $\mathbf{1}'B = \mathbf{1}'$.

We call a matrix A *consensual*, if it has equal row vectors. The largest and smallest eigenvalues in modulus of a matrix A are denoted as $\lambda_{\max}\{A\}$ and $\lambda_{\min}\{A\}$, respectively. For any matrix $A \in \mathbb{R}^{n\times n}$, we use $\mathrm{diag}(A)$ to denote its diagonal vector, i.e. $\mathrm{diag}(A) = (a_{11}, \ldots, a_{nn})'$. For any vector $u \in \mathbb{R}^n$ we use $\mathrm{Diag}(u)$ to denote the diagonal matrix with the vector u on its diagonal.

Given a vector $\pi \in \mathbb{R}^m$ with positive entries $\pi_1, \ldots, \pi_m$, the π-weighted inner product and π-weighted norm are defined, respectively, as follows:

$$\langle \mathbf{u}, \mathbf{v}\rangle_\pi = \sum_{i=1}^m \pi_i \langle u_i, v_i\rangle \quad \text{and} \quad \|\mathbf{u}\|_\pi = \sqrt{\sum_{i=1}^m \pi_i \|u_i\|^2},$$

where $\mathbf{u} := \begin{bmatrix} u_1' \\ \vdots \\ u_m' \end{bmatrix}, \mathbf{v} := \begin{bmatrix} v_1' \\ \vdots \\ v_m' \end{bmatrix} \in \mathbb{R}^{m\times n}$, and $u_i, v_i \in \mathbb{R}^n$.

When $\pi = \mathbf{1}$, we simply write $\|\mathbf{u}\|$, for which we have:

$$\frac{1}{\sqrt{\max(\pi)}}\|\mathbf{u}\|_\pi \le \|\mathbf{u}\| \le \frac{1}{\sqrt{\min(\pi)}}\|\mathbf{u}\|_\pi. \tag{1}$$

Furthermore, using the Cauchy–Schwarz inequality, we obtain:

$$\langle \mathbf{u}, \mathbf{v}\rangle_\pi = \left|\sum_{i=1}^m \pi_i \langle u_i, v_i\rangle\right| \le \sum_{i=1}^m \pi_i |\langle u_i, v_i\rangle| \le \sum_{i=1}^m \pi_i \|u_i\|\|v_i\|$$

$$\le \sqrt{\left(\sum_{i=1}^m \pi_i\|u_i\|^2\right)\left(\sum_{i=1}^m \pi_i\|v_i\|^2\right)} = \|\mathbf{u}\|_\pi \|\mathbf{v}\|_\pi. \tag{2}$$

Thus, the Cauchy–Schwarz inequality holds for the π-weighted inner product and the π-weighted norm.

A mapping $g : E \to E$ is said to be *strongly monotone* on a set $Q \subseteq E$ with the constant $\mu > 0$, if $\langle g(u) - g(v), u - v\rangle \ge \mu\|u - v\|^2$ for any $u, v \in Q$, where $\|\cdot\|$ is the corresponding norm in E. A mapping $g : E \to E$ is said to be *Lipschitz continuous* on a set $Q \subseteq E$ with the constant $L > 0$, if $\|g(u) - g(v)\| \le L\|u - v\|$.

We let $[m]$ denote the set $\{1, \ldots, m\}$ for an integer $m \ge 1$. A directed graph $\mathbb{G} = ([m], \mathsf{E})$ is specified by the set of edges $\mathsf{E} \subseteq [m] \times [m]$ of ordered pairs of nodes. Given two distinct nodes $j, l \in [m]$ ($j \neq l$), a *directed path* from node j to node l in the graph $\mathbb{G}$ is a finite (ordered) sequence of edges $\{(i_0, i_1), \ldots, (i_{t-1}, i_t)\}$ passing through distinct nodes, where $i_0 = j,\ i_t = l$, and $(i_{s-1}, i_s) \in \mathsf{E}$ for all $s = 1, \ldots, t$.

Definition 1 (Graph Connectivity)**.** *A directed graph is strongly connected if there is a directed path from any node to all the other nodes in the graph.*

Given a directed path, the length of the path is the number of edges in the path.

Definition 2 (Graph Diameter)**.** *The diameter of a strongly connected directed graph $\mathbb{G}$ is the length of the longest path in the collection of all shortest directed paths connecting all ordered pairs of distinct nodes in $\mathbb{G}$.*

We denote the diameter of the graph $\mathbb{G}$ by $\mathsf{D}(\mathbb{G})$.

In what follows, we consider special collections of shortest directed paths which we refer to as shortest-path covering of the graph. Let $\mathbf{p}_{jl}$ denote a *shortest directed path from node j to node l*, where $j \neq l$.

Definition 3 (Shortest-Path Graph Covering)**.** *A collection $\mathcal{P}$ of directed paths in $\mathbb{G}$ is a shortest-path graph covering if $\mathbf{p}_{jl} \in \mathcal{P}$ and $\mathbf{p}_{lj} \in \mathcal{P}$ for any two nodes $j,\ l \in [m],\ j \neq l$.*

Denote by $\mathcal{S}(\mathbb{G})$ the collection of all possible shortest-path coverings of the graph $\mathbb{G}$.

Given a shortest-path covering $\mathcal{P} \in \mathcal{S}(\mathbb{G})$ and an edge $(j, l) \in \mathsf{E}$, the *utility of the edge (j, l)* with respect to the covering $\mathcal{P}$ is the number of shortest paths in $\mathcal{P}$ that pass through the edge (j, l). Define $\mathsf{K}(\mathcal{P})$ as the maximum edge-utility in $\mathcal{P}$ taken over all edges in the graph, i.e.,

$$\mathsf{K}(\mathcal{P}) = \max_{(j,l)\in\mathsf{E}} \sum_{\mathbf{p}\in\mathcal{P}} \chi_{\{(j,l)\in\mathbf{p}\}},$$

where $\chi_{\{(j,l)\in\mathbf{p}\}}$ is the indicator function taking value 1 when $(j, l) \in \mathbf{p}$ and, otherwise, taking value 0.

Definition 4 (Maximal Edge-Utility)**.** *Let $\mathbb{G} = ([m], \mathsf{E})$ be a strongly connected directed graph. The maximal edge-utility in the graph $\mathbb{G}$ is the maximum value of $\mathsf{K}(\mathcal{P})$ taken over all possible shortest-path coverings $\mathcal{P} \in \mathcal{S}(\mathbb{G})$, i.e.,*

$$\mathsf{K}(\mathbb{G}) = \max_{\mathcal{P}\in\mathcal{S}(\mathbb{G})} \mathsf{K}(\mathcal{P}).$$

As an example, consider a directed-cycle graph $\mathbb{G}$ of the nodes $1, 2, \ldots, m$. Then, $\mathsf{K}(\mathbb{G}) = m - 1$.

Given a directed graph $\mathbb{G} = ([m], \mathsf{E})$, we define the in-neighbor and out-neighbor set for every agent i, as follows:

$$\mathcal{N}_i^{\mathrm{in}} = \{j \in [m] \mid (j, i) \in \mathsf{E}\},$$

$$\mathcal{N}_i^{\mathrm{out}} = \{\ell \in [m] \mid (i, \ell) \in \mathsf{E}\}.$$

When the graph varies over time, we use a subscript to indicate the time instance. For example, E_k will denote the edge-set of a graph $\mathbb{G}_k$, $\mathcal{N}_{ik}^{\mathrm{in}}$ and $\mathcal{N}_{ik}^{\mathrm{out}}$ denote the in-neighbors and the out-neighbors of a node i, respectively. In our setting here, the agents will be the nodes in the graph, so we will use the terms "node" and "agent" interchangeably.

III. Problem Formulation

We consider a non-cooperative game between m agents. For each agent $i \in [m]$, let $J_i(\cdot)$ and $X_i \subseteq \mathbb{R}^{n_i}$ be the cost function and the action set of the agent. Let $n = \sum_{i=1}^m n_i$ be the size of the joint action vector of the agents. Each function $J_i(x_i, x_{-i})$ depends on x_i and x_{-i}, where $x_i \in X_i$ is the action of the agent i and $x_{-i} \in X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_m$ denotes the joint action of all agents except agent i.

Denote the game by $\Gamma = ([m], \{J_i\}, \{X_i\})$. A solution to the game Γ is a Nash equilibrium (NE) $x^* \in X_1 \times \cdots \times X_m$ such that for every agent $i \in [m]$, we have:

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*), \quad \forall x_i \in X_i. \tag{3}$$

When for every agent i, the action set X_i is closed and convex, and the cost function $J_i(x_i, x_{-i})$ is also convex and differentiable in x_i for each $x_{-i} \in X_{-i}$, a NE $x^* \in X$ of the game can be alternatively characterized by using the first-order optimality conditions. Specifically, $x^* \in X$ is a NE of the game if and only if for all $i \in [m]$, we have:

$$\langle \nabla_i J_i(x_i^*, x_{-i}^*), x_i - x_i^* \rangle \geq 0, \quad \forall x_i \in X_i. \tag{4}$$

Using the Euclidean projection property, it can be seen that the preceding relation is equivalent to:

$$x_i^* = \Pi_{X_i}[x_i^* - \alpha_i \nabla_i J_i(x_i^*, x_{-i}^*)], \quad \forall i \in [m], \tag{5}$$

where $\alpha_i > 0$ is an arbitrary scalar. By stacking the relations in (5), we can rewrite them in a compact form. By this way, x^* is a NE for the game Γ if and only if:

$$x^* = \Pi_X[x^* - F_\alpha(x^*)], \tag{6}$$

where $X = X_1 \times \cdots \times X_m$ is the agents' joint action set and $F_\alpha(\cdot)$ is the scaled gradient mapping of the game, defined by

$$F_\alpha(x) \triangleq \begin{bmatrix} \alpha_1 \nabla_1 J_1(x_1, x_{-1}) \\ \vdots \\ \alpha_m \nabla_m J_m(x_m, x_{-m}) \end{bmatrix}, \tag{7}$$

where $\nabla_i J_i(x_i, x_{-i}) = \nabla_{x_i} J_i(x_i, x_{-i})$ for all $i \in [m]$.

In the absence of constraints on the agents' access to each others' actions, an NE point can be computed by implementing a simple iterative algorithm (see [8]). In particular, starting with some initial point $x_i^0 \in X_i$, each agent i updates its decision at time k as follows:

$$x_i^{k+1} = \Pi_{X_i}[x_i^k - \alpha_i \nabla_i J_i(x_i^k, x_{-i}^k)]. \tag{8}$$

This algorithm is guaranteed to converge to a NE under suitable conditions. However, it requires that every agent i has access to all other agents' decisions x_{-i}^k at every time k.

A. Graph-constrained Agents' Interactions

In this paper, we focus on the setting where the agents' interactions over time are constrained by a sequence of directed time-varying communication graphs. When the agents interact at time k, their interactions are constrained by a directed graph $\mathbb{G}_k = ([m], \mathsf{E}_k)$, where the set of nodes is the agent set $[m]$ and E_k is the set of directed links. The directed link (j, i) indicates that agent i can receive information from agent j.

Given that our game Γ has constraints on agents' access to actions of other agents, we consider an adaptation of the basic algorithm (8) that will obey the information access as dictated by the graph $\mathbb{G}_k$ at time k. In the absence of the access to x_{-i}^k, agent i will use an estimate $z_{i,-i}^k$ instead, which leads to the following update rule for each agent $i \in [m]$:

$$x_i^{k+1} = \Pi_{X_i}[v_{ii}^{k+1} - \alpha_i \nabla_i J_i(v_{ii}^{k+1}, z_{i,-i}^{k+1})]. \tag{9}$$

where the vector v_{ii}^{k+1} is some estimate (based on what neighbors think the actions of agent i are) and $z_{i,-i}^{k+1} = (z_{i1}^{k+1}, \ldots, z_{i,i-1}^{k+1}, z_{i,i+1}^{k+1}, \ldots, z_{im}^{k+1}) \in \mathbb{R}^{n-n_i}$ consisting of the estimates z_{ij}^{k+1} that agent i has about the true decision vector x_j^k for every agent $j \neq i$. Notice that we have the index $k+1$ for the estimates v_{ii}^{k+1} and $z_{i,-i}^{k+1}$ since they are constructed at time $k+1$ upon information exchange among the neighbors in the graph $\mathbb{G}_k$.

In this situation, v_{ii}^{k+1} need not belong to the set X_i at any time k, as the other agents may not know this set. Also, as agent i does not know the action space X_{-i}, the estimate z_{ij}^{k+1} need not lie in the set X_{-i}. Thus, the function $J_i(x_i, x_{-i})$ should be defined on the set $\mathbb{R}^n$, where $n = \sum_{i=1}^m n_i$. Specifically, regarding the agents' cost functions and their action sets, we use the following assumptions:

Assumption 1. *Consider the game* $\Gamma = ([m], \{J_i\}, \{X_i\})$, *and assume that for all* $i \in [m]$*:*

(a) *The mapping* $\nabla_i J_i(x_i, \cdot)$ *is Lipschitz continuous on* $\mathbb{R}^{n-n_i}$ *for every* $x_i \in \mathbb{R}^{n_i}$ *with a uniform constant* $L_{-i} > 0$.
(b) *The mapping* $\nabla_i J_i(\cdot, x_{-i})$ *is Lipschitz continuous on* $\mathbb{R}^{n_i}$ *for every* $x_{-i} \in \mathbb{R}^{n-n_i}$ *with a uniform constant* $L_i > 0$.
(c) *The mapping* $\nabla_i J_i(\cdot, x_{-i})$ *is strongly monotone on* $\mathbb{R}^{n_i}$ *for every* $x_{-i} \in \mathbb{R}^{n-n_i}$ *with a uniform constant* $\mu_i > 0$.
(d) *The set* X_i *is nonempty, convex, and closed.*

Remark 1. *Under Assumption 1, a NE point exists and it is unique (Theorem 2.3.3 of [8]). Moreover, it can be equivalently captured as the fixed point solution (see* (6)*). The differentiability of the cost functions on a larger range of* $x_{-i} \in \mathbb{R}^{n-n_i}$ *is assumed to ensure that the algorithm* (9) *is well defined.*

IV. Distributed Algorithm

We consider the distributed algorithm over a sequence $\{\mathbb{G}_k\}$ of underlying directed communication graphs. We assume that every node has a self-loop in each graph $\mathbb{G}_k$, so that the neighbor sets $\mathcal{N}_{ik}^{\text{in}}$ and $\mathcal{N}_{ik}^{\text{out}}$ contain agent i at all times. Specifically, we use the following assumption.

Assumption 2. *Each graph* $\mathbb{G}_k = ([m], \mathsf{E}_k)$ *is strongly connected and has a self-loop at every node* $i \in [m]$.

For each k, each agent i has a column vector $z_i^k = (z_{i1}^k, \ldots, z_{im}^k)' \in \mathbb{R}^n$, where z_{ij}^k is agent i's estimate of the decision x_j^k for agent $j \neq i$, while $z_{ii}^k = x_i^k$. Let $z_{j,-i}^k = (z_{j1}^k, \ldots, z_{j,i-1}^k, z_{j,i+1}^k, \ldots, z_{jm}^k)' \in \mathbb{R}^{n-n_i}$ be the estimate of agent j without the i-th block-component. Hence, z_i^k consists of the decision z_{ii}^k of agent i and the estimate $z_{i,-i}^k$ of agent i for the decisions of the other agents.

At time k, every agent i sends z_i^k to its out-neighbors $\ell \in \mathcal{N}_{ik}^{\text{out}}$ and receives estimates z_j^k from its in-neighbors $j \in \mathcal{N}_{ik}^{\text{in}}$. Once the information is exchanged, agent i computes v_{ii}^{k+1} that is an estimate of x_i^k based on what the in-neighbors think the actions of agent i is, and, the estimate $z_{i,-i}^{k+1}$. Then, agent i updates its own action accordingly. Intuitively, using the estimates based on the information gathered from neighbors can improve the accuracy of the estimates including the estimate of its own action since more information is taken into account. The agents' estimates are constructed by using a row-stochastic weight matrix W_k that is compliant with the connectivity structure of the graph $\mathbb{G}_k$, in the sense that:

$$\begin{cases} [W_k]_{ij} > 0, & \text{when} \quad j \in \mathcal{N}_{ik}^{\text{in}}, \\ [W_k]_{ij} = 0, & \text{otherwise.} \end{cases} \tag{10}$$

Note that every agent i controls the entries in the ith row of W_k, which does not require any coordination of the weights among the agents. In fact, balancing the weights [6] would require some coordination among the agents or some side information about the structure of the graphs $\mathbb{G}_k$, which we avoid imposing in this paper.

The estimate v_{ii}^{k+1} of x_i^{k+1} is constructed based on the information that i receives from its in-neighbors $j \in \mathcal{N}_{ik}^{\text{in}}$ with the corresponding weights, as follows:

$$v_{ii}^{k+1} = \sum_{l=1}^m [W_k]_{il} z_{li}^k. \tag{11}$$

Agent i estimate of other agents' actions is computed as:

$$z_{i,-i}^{k+1} = \sum_{j=1}^m [W_k]_{ij} z_{j,-i}^k. \tag{12}$$

Finally, using these estimates, agent i updates its own action according to the following formula

$$x_i^{k+1} = \Pi_{X_i}[v_{ii}^k - \alpha_i \nabla_i J_i(v_{ii}^{k+1}, z_{i,-i}^{k+1})].$$

The procedure is summarized in Algorithm 1.

Algorithm 1: Distributed Method

Every agent $i \in [m]$ selects a stepsize $\alpha_i > 0$ and an arbitrary initial vector $z_i^0 \in \mathbb{R}^n$.
for $k = 0, 1, \ldots,$ every agent $i \in [m]$ does the following:
- Receives z_j^k from in-neighbors $j \in \mathcal{N}_{ik}^{\text{in}}$;
- Sends z_i^k to out-neighbors $\ell \in \mathcal{N}_{ik}^{\text{out}}$;
- Chooses the weights $[W_k]_{ij}, j \in [m]$;
- Computes the estimates v_{ii}^{k+1} and $z_{i,-i}^{k+1}$ by
  $v_{ii}^{k+1} = \sum_{l=1}^m [W_k]_{il} z_{li}^k$, and
  $z_{i,-i}^{k+1} = \sum_{j=1}^m [W_k]_{ij} z_{j,-i}^k$;
- Updates action x_i^{k+1} by
  $x_i^{k+1} = \Pi_{X_i}[v_{ii}^{k+1} - \alpha_i \nabla_i J_i(v_{ii}^{k+1}, z_{i,-i}^{k+1})]$;
- Updates the estimate z_{ii}^{k+1} by $z_{ii}^{k+1} = x_i^{k+1}$;

end for.

We make the following assumption on the matrices W_k.

Assumption 3. *For each* $k \geq 0$*, the weight matrix* W_k *is row-stochastic and compatible with the graph* $\mathbb{G}_k$ *i.e., it satisfies relation* (10)*. Moreover, there exist a scalar* $w > 0$ *such that* $\min(W_k^+) \geq w$ *for all* $k \geq 0$.

V. Basic results

In this section, we provide some basic results related to norms of linear combinations of vectors, graphs, stochastic matrices, and the gradient method.

A. Linear Combinations and Graphs

Since the mixing terms $\sum_{j=1}^m [W_k]_{ij} z_j^k$ used in Algorithm 1 are special linear combination of z_j^k, we start by establishing a result for linear combinations of vectors. In particular, in the forthcoming lemma, we provide a relation for the squared norm of a linear combination of vectors, which will be used in our analysis with different identifications.

Lemma 1. *Let $\{u_i,\, i \in [m]\} \subset \mathbb{R}^n$ be a collection of m vectors and $\{\gamma_i,\, i \in [m]\}$ be a collection of m scalars. Then, the following statements are valid:*
(a) We have:

$$\left\|\sum_{i=1}^m \gamma_i u_i\right\|^2 = \left(\sum_{j=1}^m \gamma_j\right)\sum_{i=1}^m \gamma_i \|u_i\|^2 - \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \gamma_i\gamma_j\|u_i - u_j\|^2.$$

(b) If $\sum_{i=1}^m \gamma_i = 1$ holds, then for all $u \in \mathbb{R}^n$ we have:

$$\left\|\sum_{i=1}^m \gamma_i u_i - u\right\|^2 = \sum_{i=1}^m \gamma_i \|u_i - u\|^2 - \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \gamma_i\gamma_j\|u_i - u_j\|^2.$$

Proof. (a) For $\|\sum_{i=1}^m \gamma_i u_i\|^2$, we have:

$$\begin{aligned}\left\|\sum_{i=1}^m \gamma_i u_i\right\|^2 &= \langle \sum_{i=1}^m \gamma_i u_i, \sum_{j=1}^m \gamma_j u_j\rangle \\ &= \sum_{i=1}^m\sum_{j=1}^m \gamma_i\gamma_j\langle u_i, u_j\rangle \\ &= \sum_{i=1}^m \gamma_i^2\|u_i\|^2 + \sum_{i=1}^m\sum_{j=1, j\neq i}^m \gamma_i\gamma_j\langle u_i, u_j\rangle.\end{aligned}$$

Using the identity

$$2\langle v, w\rangle = \|v\|^2 + \|w\|^2 - \|v - w\|^2,$$

which is valid for any two vectors v and w, we obtain:

$$\begin{aligned}\left\|\sum_{i=1}^m \gamma_i u_i\right\|^2 = &\sum_{i=1}^m \gamma_i^2\|u_i\|^2 \\ &+ \frac{1}{2}\sum_{i=1}^m\sum_{j=1, j\neq i}^m \gamma_i\gamma_j\left(\|u_i\|^2 + \|u_j\|^2 - \|u_i - u_j\|^2\right).\end{aligned} \tag{13}$$

Note that

$$\begin{aligned}&\sum_{i=1}^m\sum_{j=1, j\neq i}^m \gamma_i\gamma_j\left(\|u_i\|^2 + \|u_j\|^2\right) \\ &= \sum_{i=1}^m\left(\sum_{j=1, j\neq i}^m \gamma_j\right)\gamma_i\|u_i\|^2 + \sum_{j=1}^m\left(\sum_{i=1, i\neq j}^n \gamma_i\right)\gamma_j\|u_j\|^2,\end{aligned}$$

we further obtain that

$$\sum_{i=1}^m\sum_{j=1, j\neq i}^m \gamma_i\gamma_j\left(\|u_i\|^2 + \|u_j\|^2\right) = 2\sum_{i=1}^m\left(\sum_{j=1, j\neq i}^m \gamma_j\right)\gamma_i\|u_i\|^2.$$

Therefore, by substituting the preceding equality in relation (13) we find that:

$$\begin{aligned}\left\|\sum_{i=1}^m \gamma_i u_i\right\|^2 &= \sum_{i=1}^m \gamma_i^2\|u_i\|^2 + \sum_{i=1}^m\left(\sum_{j=1, j\neq i}^m \gamma_j\right)\gamma_i\|u_i\|^2 \\ &\quad - \frac{1}{2}\sum_{i=1}^m\sum_{j=1, j\neq i}^m \gamma_i\gamma_j\|u_i - u_j\|^2.\end{aligned}$$

The first two terms in the preceding relation give:

$$\sum_{i=1}^m \gamma_i^2\|u_i\|^2 + \sum_{i=1}^m\left(\sum_{j=1, j\neq i}^m \gamma_j\right)\gamma_i\|u_i\|^2 = \sum_{i=1}^m\left(\sum_{j=1}^m \gamma_j\right)\gamma_i\|u_i\|^2,$$

implying that:

$$\left\|\sum_{i=1}^m \gamma_i u_i\right\|^2 = \sum_{i=1}^m\left(\sum_{j=1}^m \gamma_j\right)\gamma_i\|u_i\|^2 - \frac{1}{2}\sum_{i=1}^m\sum_{j=1, j\neq i}^m \gamma_i\gamma_j\|u_i - u_j\|^2. \tag{14}$$

The relation in part (a) follows by noting that the sum $\sum_{i=1}^m\sum_{j=1, j\neq i}^m \gamma_i\gamma_j\|u_i - u_j\|^2$ does not change when we add the terms $\gamma_i\gamma_j\|u_i - u_j\|^2$ for $j = i$, since they are all zero.
(b) Suppose now that $\sum_{i=1}^m \gamma_i = 1$. Then, for any vector $u \in \mathbb{R}^n$, we have:

$$\sum_{i=1}^m \gamma_i u_i - u = \sum_{i=1}^m \gamma_i u_i - \left(\sum_{i=1}^m \gamma_i\right)u = \sum_{i=1}^m \gamma_i(u_i - u).$$

We apply the relation from part (a) where u_i is replaced with $u_i - u$, and by using the fact that $\sum_{i=1}^m \gamma_i = 1$, then for all $u \in \mathbb{R}^n$, we obtain:

$$\left\|\sum_{i=1}^m \gamma_i u_i - u\right\|^2 = \sum_{i=1}^m \gamma_i\|u_i - u\|^2 - \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \gamma_i\gamma_j\|u_i - u_j\|^2.$$

■

We have the following result as an immediate consequence of Lemma 1(b).

Corollary 1. *Choosing $u = \sum_{j=1}^m \gamma_\ell u_\ell$ in Lemma 1(b) yields, in particular, that*

$$\frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \gamma_i\gamma_j\|u_i - u_j\|^2 = \sum_{i=1}^m \gamma_i\left\|u_i - \sum_{\ell=1}^m \gamma_\ell u_\ell\right\|^2. \tag{15}$$

Substituting the preceding relation back in the relation in part (b) of Lemma 1 gives for all $u \in \mathbb{R}^n$,

$$\left\|\sum_{i=1}^m \gamma_i u_i - u\right\|^2 = \sum_{i=1}^m \gamma_i\|u_i - u\|^2 - \sum_{i=1}^m \gamma_i\left\|u_i - \sum_{\ell=1}^m \gamma_\ell u_\ell\right\|^2. \tag{16}$$

Relation (16) *is valid when $\sum_{i=1}^m \gamma_i = 1$. If additionally, the scalars γ_i are non-negative, then relation* (16) *coincides with the well known relation for weighted averages of vectors.*

There are certain contraction properties of the distributed method, which are inherited from the use of the mixing term $\sum_{j=1}^m [W_k]_{ij} z_j^k$, and the fact that the matrix W_k is compliant with a directed strongly connected graph $\mathbb{G}_k$. Lemma 1 provides a critical result that will help us capture these contraction properties. However, Lemma 1 alone is not sufficient since it does not make any use of the structure of the matrix W_k related to the underlying graph $\mathbb{G}_k$.

The graph structure is exploited in the forthcoming lemma for a generic graph. More specifically, the lemma provides an important lower bound on the quantity $\sum_{(j,\ell)\in\mathsf{E}}\|z_j - z_\ell\|^2$ for a given directed graph $\mathbb{G} = ([m],\mathsf{E})$, where $z_i \in \mathbb{R}^n$ is a vector associated with a node i. The lower bound will be applied to the graph $\mathbb{G}_k$ at time k, which provides the second critical step leading us toward the contraction properties of the iterate sequences.

Lemma 2. *Let $\mathbb{G} = ([m],\mathsf{E})$ be a strongly connected directed graph, where a vector $z_i \in \mathbb{R}^n$ is associated with node i for all $i \in [m]$. Let $\mathcal{P}^* \in \mathcal{S}(\mathbb{G})$ be a shortest path covering of the graph $\mathbb{G}$. We then have:*

$$\sum_{(j,\ell)\in\mathsf{E}}\|z_j - z_\ell\|^2 \geq \frac{2}{\mathsf{D}(\mathbb{G})\mathsf{K}(\mathbb{G})}\sum_{j=1}^{m}\sum_{\ell=j+1}^{m}\|z_j - z_\ell\|^2,$$

where $\mathsf{D}(\mathbb{G})$ is the diameter of the graph and $\mathsf{K}(\mathbb{G})$ is the maximal edge-utility in the graph (see Definitions 2 and 4).

Proof. Let $\mathcal{P}^* = \{\mathbf{p}^*_{j\ell} \mid j,\ell \in [m], j \neq \ell\}$, where $\mathbf{p}^*_{j\ell}$ is a shortest path from node j to node ℓ. Let E^* be the collection of all directed links that are traversed by any path in the shortest path covering $\mathcal{P}^*$, i.e.,

$$\mathsf{E}^* = \{(i,q) \in \mathsf{E} \mid (i,q) \in \mathbf{p}^*_{j\ell} \text{ for some } \mathbf{p}^*_{j\ell} \in \mathcal{P}^*\}.$$

By the definition the maximal edge-utility $\mathsf{K}(\mathbb{G})$ (see Definition 4), we have:

$$\mathsf{K}(\mathbb{G}) \geq \mathsf{K}(\mathcal{P}^*),$$

where $\mathsf{K}(\mathcal{P}^*)$ is the maximal edge-utility with respect to the shortest path covering $\mathcal{P}^*$, i.e.,

$$\mathsf{K}(\mathcal{P}^*) = \max_{(i,q)\in\mathsf{E}}\sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}}.$$

Note that $\sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}} = 0$ when a link (i,q) is not used by any of the paths in $\mathcal{P}^*$. Thus, the value $\mathsf{K}(\mathcal{P}^*)$ is equivalently given by:

$$\mathsf{K}(\mathcal{P}^*) = \max_{(i,q)\in\mathsf{E}^*}\sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}}.$$

Consider the quantity $\mathsf{K}(\mathcal{P}^*)\sum_{(\ell,j)\in\mathsf{E}}\|z_j - z_\ell\|^2$. Since $\mathsf{E}^* \subseteq \mathsf{E}$, it follows that

$$\mathsf{K}(\mathcal{P}^*)\sum_{(j,\ell)\in\mathsf{E}}\|z_j - z_\ell\|^2 \geq \mathsf{K}(\mathcal{P}^*)\sum_{(i,q)\in\mathsf{E}^*}\|z_i - z_q\|^2 = \sum_{(i,q)\in\mathsf{E}^*}\mathsf{K}(\mathcal{P}^*)\|z_i - z_q\|^2.$$

Since $\mathsf{K}(\mathcal{P}^*)$ the maximal edge-utility with respect to paths in the collection $\mathcal{P}^*$, it follows that $\mathsf{K}(\mathcal{P}^*) \geq \sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}}$ for any link (i,q) which is used by a path in $\mathcal{P}^*$. Hence,

$$\mathsf{K}(\mathcal{P}^*)\sum_{(j,\ell)\in\mathsf{E}}\|z_j - z_\ell\|^2 \geq \sum_{(i,q)\in\mathsf{E}^*}\left(\sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}}\right)\|z_i - z_q\|^2. \quad (17)$$

Note that the sum on the right hand side in the preceding relation is taken over all links in E^* with the multiplicity with which every (i,q) is used in the shortest path covering $\mathcal{P}^*$. Thus, it can be written in terms of the paths in $\mathcal{P}^*$ which are connecting distinct nodes $j,\ell \in [m]$, as follows:

$$\sum_{(i,q)\in\mathsf{E}^*}\left(\sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}}\right)\|z_i - z_q\|^2 = \sum_{j=1}^{m}\sum_{\ell=j+1}^{m}\sum_{(i,q)\in\mathbf{p}^*_{j\ell}}\|z_i - z_q\|^2 + \sum_{j=1}^{m}\sum_{\ell=j+1}^{m}\sum_{(i,q)\in\mathbf{p}^*_{\ell j}}\|z_i - z_q\|^2. \quad (18)$$

Using the convexity of the squared norm, we have:

$$\frac{1}{|\mathbf{p}^*_{j\ell}|}\sum_{(i,q)\in\mathbf{p}^*_{j\ell}}\|z_i - z_q\|^2 \geq \left\|\frac{1}{|\mathbf{p}^*_{j\ell}|}\sum_{(i,q)\in\mathbf{p}^*_{j\ell}}(z_i - z_q)\right\|^2 = \frac{1}{|\mathbf{p}^*_{j\ell}|^2}\|z_j - z_\ell\|^2,$$

where $|\mathbf{p}^*_{j\ell}|$ denotes the length of the path $|\mathbf{p}^*_{j\ell}|$. Hence,

$$\sum_{(i,q)\in\mathbf{p}^*_{j\ell}}\|z_i - z_q\|^2 \geq \frac{1}{|\mathbf{p}^*_{j\ell}|}\|z_j - z_\ell\|^2,$$

which in view of (18) implies that

$$\sum_{(i,q)\in\mathsf{E}^*}\left(\sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}}\right)\|z_i - z_q\|^2 \geq \sum_{j=1}^{m}\sum_{\ell=j+1}^{m}\frac{1}{|\mathbf{p}^*_{j\ell}|}\|z_j - z_\ell\|^2 + \sum_{j=1}^{m}\sum_{\ell=j+1}^{m}\frac{1}{|\mathbf{p}^*_{\ell j}|}\|z_j - z_\ell\|^2.$$

Hence, by the definition of the graph diameter $\mathsf{D}(\mathbb{G})$, we have $|\mathbf{p}^*_{\ell j}| \leq \mathsf{D}(\mathbb{G})$ for any $j \neq \ell$, implying that

$$\sum_{(i,q)\in\mathsf{E}^*}\left(\sum_{\mathbf{p}\in\mathcal{P}^*}\chi_{\{(i,q)\in\mathbf{p}\}}\right)\|z_i - z_q\|^2 \geq \frac{2}{\mathsf{D}(\mathbb{G})}\sum_{j=1}^{m}\sum_{\ell=j+1}^{m}\|z_j - z_\ell\|^2. \quad (19)$$

Using the relations in (17) and (19), we obtain

$$\sum_{(j,\ell)\in\mathsf{E}}\|z_j - z_\ell\|^2 \geq \frac{2}{\mathsf{D}(\mathbb{G})\mathsf{K}(\mathcal{P}^*)}\sum_{j=1}^{m}\sum_{\ell=j+1}^{m}\|z_j - z_\ell\|^2.$$

The desired relation follows by using the fact $\mathsf{K}(\mathbb{G}) \geq \mathsf{K}(\mathcal{P}^*)$. ■

B. Implications of Stochastic Nature of Matrix W_k

Here, we focus on the matrix sequence $\{W_k\}$ and provide some basic relations due to the row-stochasticity of the matrices. In the forthcoming lemma, we state some convergence properties of the transition matrices $W_kW_{k-1}\cdots W_t$, which are valid under the assumptions of strong connectivity of the graphs $\mathbb{G}_k$ and the graph compatibility of the matrices W_k. These properties are known for such a sequence of matrices (see Lemma 2 in [14]).

Lemma 3. *Let Assumption 2 hold, and let $\{W_k\}$ be a matrix sequence satisfying Assumption 3. Then, we have*

(a) *There exists a sequence $\{\pi_k\}$ of stochastic vectors such that $\pi'_{k+1}W_k = \pi'_k$ for all $k \geq 0$.*

(b) *The entries of each π_k have a uniform lower bound, i.e.,*

$$[\pi_k]_i \geq \frac{w^m}{m} \quad \textit{for all } i \in [m] \textit{ and all } k \geq 0,$$

where w is the lower bound on the positive entries of the matrices W_k from Assumption 3.

Proof. By Assumption 2, the sequence $\{W_k\}$ of row-stochastic matrices is ergodic. Thus, Theorem 4.20 of [21] on backwards products implies that there exists a unique sequence of absolute probability vectors $\{\pi_k\}$, i.e., a sequence $\{\pi_k\}$ of stochastic vectors such that $\pi'_{k+1}W_k = \pi'_k$ for all $k \geq 0$. This shows the result in part (a). The statement in part (b) follows from Lemma 4 and Remark 2 of [14]. Specifically, in the proof of Lemma 4 of [14], using the lower bound $[W_k]_{ii} \geq w$ and $B = 1$, we obtain $\delta' \geq w^m$. Remark 2 of [14] then gives $[\pi_k]_i \geq \frac{w^m}{m}$ for all $i \in [m]$ and all $k \geq 0$. ■

The stochastic vectors π_k from Lemma 3 will be used to define an appropriate Lyapunov function associated with the method. We note that the sequence $\{\pi_k\}$ is an absolute probability sequence [21], which has also been used in [25] to study the ergodicity properties of a more involved random matrix sequences.

C. Contraction Property of Gradient Method

In this section, we analyze the contraction property of the gradient mapping $F(\cdot)$ of the game when the mapping is strongly monotone and Lipschitz continuous.

Lemma 4. *Let Assumptions 1(a) and 1(b) hold. For all $x, y \in \mathbb{R}^n$ such that $x_i, y_i \in \mathbb{R}^{n_i}$ and $x_{-i}, y_{-i} \in \mathbb{R}^{n-n_i}$, we have*

$$\|\nabla_i J_i(x) - \nabla_i J_i(y)\|^2 \leq \left(L_{-i}^2 + L_i^2\right)\|x - y\|^2, \quad (20)$$

where L_{-i} and L_i are the constants from Assumptions 1(a) and 1(b), respectively.

Proof. First, we write

$$\begin{aligned}&\nabla_i J_i(x_i, x_{-i}) - \nabla_i J_i(y_i, y_{-i})\\ =&\nabla_i J_i(x_i, x_{-i}) - \nabla_i J_i(x_i, y_{-i}) + \nabla_i J_i(x_i, y_{-i}) - \nabla_i J_i(y_i, y_{-i}).\end{aligned}$$

By Assumption 1(a), we have

$$\|\nabla_i J_i(x_i, x_{-i}) - \nabla_i J_i(x_i, y_{-i})\| \leq L_{-i}\|x_{-i} - y_{-i}\|, \quad (21)$$

while Assumption 1(b) gives

$$\|\nabla_i J_i(x_i, y_{-i}) - \nabla_i J_i(y_i, y_{-i})\| \leq L_i\|x_i - y_i\|. \quad (22)$$

Therefore,

$$\begin{aligned}&\|\nabla_i J_i(x_i, x_{-i}) - \nabla_i J_i(y_i, y_{-i})\|^2\\ \leq&\frac{\xi}{\xi - 1}\|\nabla_i J_i(x_i, x_{-i}) - \nabla_i J_i(x_i, y_{-i})\|^2\\ &+ \xi\|\nabla_i J_i(x_i, y_{-i}) - \nabla_i J_i(y_i, y_{-i})\|^2\\ \leq&\frac{\xi}{\xi - 1}L_{-i}^2\|x_{-i} - y_{-i}\|^2 + \xi L_i^2\|x_i - y_i\|^2,\end{aligned}$$

where the first inequality follows from $(a+b)^2 \leq \frac{\xi}{\xi-1}a^2 + \xi b^2$, valid for any $a, b \in \mathbb{R}$ and $\xi > 1$, while the last inequality follows from (21) and (22). The result is obtained by choosing $\xi = 1 + \frac{L_{-i}^2}{L_i^2} > 1$ so that

$$\frac{\xi}{\xi - 1}L_{-i}^2 = \xi L_i^2 = L_{-i}^2 + L_i^2.$$

■

In our analysis of Algorithm 1, we use a mapping $\mathbf{F}_\alpha(\cdot) : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ to capture the updates for all agents $i \in [m]$ at any time k. Specifically, given a matrix $\mathbf{z} \in \mathbb{R}^{m\times n}$, let $\mathbf{z}_i$ be the vector in the ith row of $\mathbf{z}$. Then, the ith row of the matrix $\mathbf{F}_\alpha(\mathbf{z})$ is defined by

$$[\mathbf{F}_\alpha(\mathbf{z})]_{i:} = (\mathbf{0}'_{n_1}, \ldots, \mathbf{0}'_{n_{i-1}}, \alpha_i(\nabla_i J_i(\mathbf{z}_i))', \mathbf{0}'_{n_{i+1}}, \ldots, \mathbf{0}'_{n_m}). \quad (23)$$

Lemma 5. *Let Assumptions 1(a) and 1(b) hold. Consider the mapping $\mathbf{F}_\alpha(\cdot)$ defined by (23), where $\alpha_i > 0$ for all $i \in [m]$. Then, for any weighted norm that is defined by a stochastic vector $\pi > \mathbf{0}$, we have for all $\mathbf{z}, \mathbf{y} \in \mathbb{R}^{m\times n}$,*

$$\|\mathbf{F}_\alpha(\mathbf{z}) - \mathbf{F}_\alpha(\mathbf{y}))\|_\pi^2 \leq \mathbf{L}_\alpha^2\|\mathbf{z} - \mathbf{y}\|_\pi^2, \quad (24)$$

where $\mathbf{L}_\alpha = \sqrt{\max_{i\in[m]}\{\alpha_i^2\left(L_{-i}^2 + L_i^2\right)\}}$.

Proof. Let $\mathbf{z}, \mathbf{y} \in \mathbb{R}^{m\times n}$ be arbitrary matrices, and let $\mathbf{z}_i$ and $\mathbf{y}_i$ denote the vectors in the ith row of $\mathbf{z}$ and $\mathbf{y}$, respectively, for all $i \in [m]$. Next, we apply Lemma 4 as follows

$$\begin{aligned}&\|\mathbf{F}_\alpha(\mathbf{z}) - \mathbf{F}_\alpha(\mathbf{y})\|_\pi^2 = \sum_{i=1}^m \pi_i\alpha_i^2\|\nabla_i J_i(\mathbf{z}_i) - \nabla_i J_i(\mathbf{y}_i)\|^2\\ &\leq \sum_{i=1}^m \pi_i\alpha_i^2\left(L_{-i}^2 + L_i^2\right)\|\mathbf{z}_i - \mathbf{y}_i\|^2 \leq \mathbf{L}_\alpha^2\|\mathbf{z} - \mathbf{y}\|_\pi^2,\end{aligned}$$

where $\mathbf{L}_\alpha = \sqrt{\max_{i\in[m]}\{\alpha_i^2\left(L_{-i}^2 + L_i^2\right)\}}$. ■

VI. Convergence Analysis

In this section, under the given assumptions, we will prove that the iterate sequences $\{x_i^k\}$ generated by Algorithm 1 converge to the Nash equilibrium for all agents $i \in [m]$.

A. Contractive Property of Weighted Dispersion

We start our analysis by considering generic vectors of the form $r_i = \sum_{j=1}^m W_{ij}z_j,\ i \in [m]$, where W is an $m \times m$ row-stochastic matrix and $z_j \in \mathbb{R}^n$ for all j. Noting that each vector r_i is a convex combination of the vectors $z_j,\ j \in [m]$, we make use of Lemma 1 (and its implications) to obtain an upper bound of some weighted dispersion of the vectors $r_1, \ldots, r_m$ in terms of a weighted dispersion of the original vectors $z_1, \ldots, z_m$, as seen in the forthcoming lemma. The derivation of the upper bound also makes use of Lemma 2.

Lemma 6. *Let $\mathbb{G} = ([m], \mathsf{E})$ be a strongly connected directed graph, and let W be an $m \times m$ row-stochastic matrix that is compatible with the graph and has positive diagonal entries, i.e., $W_{ij} > 0$ when $j = i$ and $(j, i) \in \mathsf{E}$, and $W_{ij} = 0$ otherwise. Also, let ϕ be a stochastic vector such that*

$$\phi' W = \pi'.$$

Consider a collection of vectors $z_1, \ldots, z_m \in \mathbb{R}^n$ and consider the vectors r_i given by $r_i = \sum_{j=1}^m W_{ij}z_j$ for all $i \in [m]$. Then, for all $u \in \mathbb{R}^n$, we have

$$\begin{aligned}\sum_{i=1}^m \phi_i\|r_i - u\|^2 \leq& \sum_{j=1}^m \pi_j\|z_j - u\|^2\\ &- \frac{\min(\phi)\left(\min(W^+)\right)^2}{\max^2(\pi)\mathsf{D}(\mathbb{G})\mathsf{K}(\mathbb{G})}\sum_{j=1}^m \pi_j\|z_j - \hat{z}_\pi\|^2,\end{aligned}$$

where $\hat{z}_\pi = \sum_{i=1}^m \pi_i z_i$, *and* $\mathsf{D}(\mathbb{G})$ *and* $\mathsf{K}(\mathbb{G})$ *are the diameter and the maximal edge-utility of the graph* $\mathbb{G}$, *respectively.*

Proof. Let $u \in \mathbb{R}^n$ be arbitrary, we first estimate $\sum_{i=1}^m \pi_i \|r_i - u\|^2$. By the definition of r_i, we have

$$\|r_i - u\|^2 = \left\| \sum_{j=1}^m W_{ij} z_j - u \right\|^2 .$$

Since W is row-stochastic, we can express the term $\|\sum_{j=1}^m W_{ij} z_j - u\|^2$ by using Lemma 1(b). Letting $\gamma = (\gamma_1, \ldots, \gamma_m)$, we apply Lemma 1(b) with $u_j = z_j$ for all j, and $\gamma = W_{i:}$, where $W_{i:}$ is the ith row-vector of the matrix W. Thus, we obtain

$$\left\| \sum_{j=1}^m W_{ij} z_j - u \right\|^2 = \sum_{j=1}^m W_{ij} \|z_j - u\|^2 - \frac{1}{2} \sum_{j=1}^m \sum_{\ell=1}^m W_{ij} W_{i\ell} \|z_j - z_\ell\|^2 .$$

We multiply the preceding inequality by ϕ_i and, then, we sum over all i to obtain

$$\sum_{i=1}^m \phi_i \|r_i - u\|^2 = \sum_{i=1}^m \phi_i \sum_{j=1}^m W_{ij} \|z_j - u\|^2 - \frac{1}{2} \sum_{i=1}^m \phi_i \sum_{j=1}^m \sum_{\ell=1}^m W_{ij} W_{i\ell} \|z_j - z_\ell\|^2.$$

In the first term on the right hand side of the preceding inequality we exchange the order of the summation and use the relation $\phi' W = \pi'$, which yields

$$\sum_{i=1}^m \phi_i \sum_{j=1}^m W_{ij} \|z_j - u\|^2 = \sum_{j=1}^m \left(\sum_{i=1}^m \phi_i W_{ij} \right) \|z_j - u\|^2 = \sum_{j=1}^m \pi_j \|z_j - u\|^2.$$

Hence,

$$\sum_{i=1}^m \phi_i \|r_i - u\|^2 = \sum_{j=1}^m \pi_j \|z_j - u\|^2 - \frac{1}{2} \sum_{i=1}^m \phi_i \sum_{j=1}^m \sum_{\ell=1}^m W_{ij} W_{i\ell} \|z_j - z_\ell\|^2. \tag{25}$$

Now, we estimate the last term in (25). To do so, we exchange the order of the summation and write

$$\begin{aligned} &\sum_{i=1}^m \phi_i \sum_{j=1}^m \sum_{\ell=1}^m W_{ij} W_{i\ell} \|z_j - z_\ell\|^2 \\ &= \sum_{j=1}^m \sum_{\ell=1}^m \|z_j - z_\ell\|^2 \left(\sum_{i=1}^m \phi_i W_{ij} W_{i\ell} \right) \\ &\geq \sum_{j=1}^m \sum_{\ell \in \mathcal{N}_j^{\text{in}}} \|z_j - z_\ell\|^2 \left(\sum_{i=1}^m \phi_i W_{ij} W_{i\ell} \right). \end{aligned}$$

Since the graph $\mathbb{G}$ is strongly connected, every node j must have a nonempty in-neighbor set $\mathcal{N}_j^{\text{in}}$. Moreover, by assumption 2, we have that $W_{jj} > 0$ every $j \in [m]$ and $W_{j\ell} > 0$ for all $\ell \in \mathcal{N}_j^{\text{in}}$. Hence,

$$\sum_{i=1}^m \pi_i W_{ij} W_{i\ell} \geq \pi_j W_{jj} W_{j\ell} \geq \pi_j \left(\min_{ij: W_{ij} > 0} W_{ij} \right)^2 > 0.$$

Therefore, using the notation $\min(\phi) = \min_{j \in [m]} \phi_j$ and $\min(W^+) = \min_{ij: W_{ij} > 0} W_{ij}$, we have

$$\begin{aligned} &\sum_{i=1}^m \pi_i \sum_{j=1}^m \sum_{\ell=1}^m W_{ij} W_{i\ell} \|z_j - z_\ell\|^2 \\ &\geq \min(\phi) \left(\min(W^+) \right)^2 \sum_{j=1}^m \sum_{\ell \in \mathcal{N}_j^{\text{in}}} \|z_j - z_\ell\|^2 \\ &= \min(\phi) \left(\min(W^+) \right)^2 \sum_{(\ell, j) \in \mathsf{E}} \|z_j - z_\ell\|^2. \end{aligned} \tag{26}$$

Next, we use Lemma 2 to bound from below the sum $\sum_{(\ell,j) \in \mathsf{E}} \|z_j - z_\ell\|^2$. Since $\mathbb{G} = ([m], \mathsf{E})$ is strongly connected directed graph, by Lemma 2 it follows that

$$\sum_{(j,\ell) \in \mathsf{E}} \|z_j - z_\ell\|^2 \geq \frac{2}{\mathsf{D}(\mathbb{G}) \mathsf{K}(\mathbb{G})} \sum_{j=1}^m \sum_{\ell=j+1}^m \|z_j - z_\ell\|^2,$$

where $\mathsf{D}(\mathbb{G})$ is the diameter of the graph $\mathbb{G}$, and $\mathsf{K}(\mathbb{G})$ is the maximal edge-utility in the graph $\mathbb{G}$. From the preceding relation and relation (26), we obtain

$$\begin{aligned} &\sum_{i=1}^m \phi_i \sum_{j=1}^m \sum_{\ell=1}^m W_{ij} W_{i\ell} \|z_j - z_\ell\|^2 \\ &\geq \frac{2 \min(\phi) \left(\min(W^+) \right)^2}{\mathsf{D}(\mathbb{G}) \mathsf{K}(\mathbb{G})} \sum_{j=1}^m \sum_{\ell=j+1}^m \|z_j - z_\ell\|^2. \end{aligned} \tag{27}$$

To get to the π-weighted average of the vectors z_j, we further write

$$\begin{aligned} \sum_{j=1}^m \sum_{\ell=j+1}^m \|z_j - z_\ell\|^2 &= \frac{1}{2} \sum_{j=1}^m \sum_{\ell=1}^m \|z_j - z_\ell\|^2 \\ &\geq \frac{1}{2 \max^2(\pi)} \sum_{j=1}^m \sum_{\ell=1}^m \pi_j \pi_\ell \|z_j - z_\ell\|^2, \end{aligned}$$

where $\max(\pi) = \max_{i \in [m]} \pi_i$. Finally, by using the weighted average-dispersion relation (15), with $\gamma_i = \phi_i$ and $u_i = x_i$ for all i, we have

$$\frac{1}{2} \sum_{j=1}^m \sum_{\ell=1}^m \pi_j \pi_\ell \|z_j - z_\ell\|^2 = \sum_{j=1}^m \phi_j \|z_j - \hat{z}_\pi\|^2,$$

where $\hat{z}_\pi = \sum_{\ell=1}^m \pi_\ell z_\ell$. Using the preceding two relations and relation (27), we see that

$$\begin{aligned} &\sum_{i=1}^m \pi_i \sum_{j=1}^m \sum_{\ell=1}^m W_{ij} W_{i\ell} \|z_j - z_\ell\|^2 \\ &\geq \frac{2 \min(\phi) \left(\min(W^+) \right)^2}{\max^2(\pi) \mathsf{D}(\mathbb{G}) \mathsf{K}(\mathbb{G})} \sum_{j=1}^m \pi_j \|z_j - \hat{z}_\pi\|^2. \end{aligned} \tag{28}$$

By substituting the estimate (28) in relation (25) we obtain

$$\sum_{i=1}^{m}\phi_i\|r_i-u\|^2\le\sum_{j=1}^{m}\pi_j\|z_j-u\|^2 - \frac{\min(\phi)\left(\min(W^+)\right)^2}{\max^2(\pi)\,\mathsf{D}(\mathbb{G})\mathsf{K}(\mathbb{G})}\sum_{j=1}^{m}\pi_j\|z_j-\hat{z}_\pi\|^2,$$

which is the desired relation. ■

Remark 2. *Let $\pi>\mathbf{0}$ and $\phi>\mathbf{0}$ in Lemma 6. Then, by applying the lemma with $r_i=\sum_{j=1}^m W_{ij}z_j$ and arbitrary vector $u\in\mathbb{R}^n$, we can write the resulting inequality in a compact form, as follows:*

$$\|W\mathbf{z}-\mathbf{1}_m u'\|_\phi^2\le\|\mathbf{z}-\mathbf{1}_m u'\|_\pi^2-\eta\|\mathbf{z}-\mathbf{1}_m\hat{z}_\pi'\|_\pi^2,$$

where $\mathbf{z}$ is an $m\times n$ matrix with z_i' in its ith row, $\hat{z}_\pi=\sum_{i=1}^m\pi_i z_i$, and

$$\eta=\frac{\min(\phi)\min(W^+)^2}{\max^2(\pi)\mathsf{D}(\mathbb{G})\mathsf{K}(\mathbb{G})}\in(0,1).$$

Remark 3. *Applying Lemma 6 to time-varying matrix W_k associated with a graph $\mathbb{G}_k$, and stochastic vectors $\pi_k>0$, $\pi_{k+1}>0$ satisfying $\pi_{k+1}'W_k=\pi_k'$ yields for arbitrary vector $u\in\mathbb{R}^n$,*

$$\|W\mathbf{z}-\mathbf{1}_m u'\|_{\pi_{k+1}}^2\le\|\mathbf{z}-\mathbf{1}_m u'\|_{\pi_k}^2-\eta_k\|\mathbf{z}-\mathbf{1}_m\hat{z}_{\pi_k}'\|_{\pi_k}^2,$$

where $\mathbf{z}$ is an $m\times n$ matrix with z_i' in its ith row $\hat{z}_{\pi_k}=\sum_{i=1}^m[\pi_k]_i z_i$, $w>0$ is the minimal positive entry in W_k, i.e., $[W_k]_{ij}\ge w$ for all i,j such that $[W_k]_{ij}>0$, and

$$\eta_k=\frac{\min(\pi_{k+1})w^2}{\max^2(\pi_k)\mathsf{D}(\mathbb{G}_k)\mathsf{K}(\mathbb{G}_k)}\in(0,1). \tag{29}$$

To deal with the time-varying graphs $\mathbb{G}_k$ and their associated matrices W_k, under suitable assumptions, we will employ Lemma 6 with the absolute probability sequence $\{\pi_k\}$ from Lemma 3, as indicated by Remark 3. To do so, we will use a lower bound on the constants η_k in (29), defined as follows:

$$\boldsymbol{\eta}=\min_{k\ge0}\eta_k,\qquad \boldsymbol{\eta}\in(0,1). \tag{30}$$

We note that $\boldsymbol{\eta}\in(0,1)$ holds under Assumption 2 and Assumption 3. Generally, the worst bounds for $\mathsf{D}(\mathbb{G}_k)$ and $\mathsf{K}(\mathbb{G}_k)$ are

$$\mathsf{D}(\mathbb{G}_k)\le m-1,\qquad \mathsf{K}(\mathbb{G}_k)\le m-1\qquad\text{for all } k\ge0.$$

By Lemma 3, the smallest positive entries of π_{k+1} can be uniformly bounded by $\frac{w^m}{m}$, while $\max(\pi_k)\le1$. This yields the following extremely pessimistic lower bound for η_k:

$$\eta_k\ge\frac{w^{m+2}}{m(m-1)^2}\qquad\text{for all } k\ge0.$$

If the graphs $\mathbb{G}_k$ are more structured, a tighter bound can be obtained. For example, if each graph $\mathbb{G}_k$ is a directed cycle over the nodes $1,\ldots,m$ and has self-loops at every node, then

$$\mathsf{D}(\mathbb{G}_k)=m-1,\qquad \mathsf{K}(\mathbb{G}_k)=m-1\qquad\text{for all } k\ge0.$$

However, in this case, each graph $\mathbb{G}_k$ is 2-regular, and if the diagonal entries of W_k are all equal to $1-w$, with $w\in(0,1)$, then we would have $\pi_k=\frac{1}{m}\mathbf{1}$ for all k, yielding

$$\eta_k\ge\frac{mw^2}{(m-1)^2}\qquad\text{for all } k\ge0.$$

By choosing $w=(m-1)/m$, we would obtain $\eta_k\ge\frac{1}{m}$ for all $k\ge0$. While exploring the graph structures leading to tighter lower bounds η_k is interesting on its own, it is not further explored in this paper. Throughout the rest of the work, we will just use the fact that a bound $\boldsymbol{\eta}\in(0,1)$ in (30) exists.

B. Convergence of Algorithm 1 over Time-varying Network

Assuming that the time-varying directed graphs $\mathbb{G}_k=([m],\mathsf{E}_k)$ are strongly connected and the weight matrices $W_k\in\mathbb{R}^{m\times m}$ are row-stochastic matrices that are compatible with the graph $\mathbb{G}_k$. Consider Algorithm 1, and let $z_i^k=(z_{i1}^k,\ldots,z_{im}^k)'\in\mathbb{R}^n$ for all $k\ge0$. Let $\{\pi_k\}$ be the sequence of stochastic vectors satisfying $\pi_{k+1}'W_k=\pi_k'$ with $\pi_k>\mathbf{0}$ for all k. Using the iterates $z_i^k=(z_{i1}^k,\ldots,z_{im}^k)'$, define matrices

$$\mathbf{z}^k=\begin{bmatrix}(z_1^k)'\\ \vdots\\ (z_m^k)'\end{bmatrix},\quad \hat{\mathbf{z}}^k=\mathbf{1}_m(\hat{z}_{\pi_k}^k)',\quad \mathbf{x}^*=\mathbf{1}_m(x^*)', \tag{31}$$

where $\hat{z}_{\pi_k}^k=\sum_{i=1}^m[\pi_k]_i z_i^k$ and x^* is an NE point of the game.

Our next result provides a basic relation for the time-varying π_k-weighted norm of the difference between $\mathbf{z}^k$ and $\mathbf{x}^*$.

Lemma 7. *Let Assumption 1, Assumption 2, and Assumption 3 hold. Consider Algorithm 1 and the notation in* (31). *We then have for all $k\ge0$,*

$$\begin{aligned}\|\mathbf{z}^{k+1}-\mathbf{x}^*\|_{\pi_{k+1}}^2\le&\left(1+\mathbf{L}_\alpha^2\right)\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}^2-2\boldsymbol{\beta}_\alpha\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_k}^2\\&+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\\&+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_{k+1}},\end{aligned}$$

where $\{\pi_k\}$ is the absolute probability sequence from Lemma 3, $\mathbf{L}_\alpha=\sqrt{\max_{i\in[m]}\{\alpha_i^2\left(L_{-i}^2+L_i^2\right)\}}$, and $\boldsymbol{\beta}_\alpha=\min_{k\ge0}\min_{i\in[m]}\{[\pi_{k+1}]_i\alpha_i\mu_i\}$.

Proof. Under Assumption 1, an NE point $x^*\in X$ exists and it is unique. According to the update formula for $\mathbf{z}^{k+1}$ from Algorithm 1, using the notation in (31), we have that

$$\begin{aligned}&\|\mathbf{z}^{k+1}-\mathbf{x}^*\|_{\pi_{k+1}}^2\\&=\sum_{i=1}^m[\pi_{k+1}]_i\left(\|x_i^{k+1}-x_i^*\|^2+\|z_{i,-i}^{k+1}-x_{-i}^*\|^2\right).\end{aligned} \tag{32}$$

Using the definition of x_i^{k+1} in Algorithm 1, the fixed point relation for the NE in (5), and the non-expansiveness property of the projection, we obtain

$$\begin{aligned}&\|x_i^{k+1}-x_i^*\|^2\\&=\|\Pi_{X_i}\left[v_{ii}^{k+1}-\alpha_i\nabla_iJ_i(v_{ii}^{k+1},z_{i,-i}^{k+1})\right]-\Pi_{X_i}[x_i^*-\alpha_i\nabla_iJ_i(x^*)]\|^2\\&\le\|v_{ii}^{k+1}-\alpha_i\nabla_iJ_i(v_{ii}^{k+1},z_{i,-i}^{k+1})-(x_i^*-\alpha_i\nabla_iJ_i(x^*))\|^2.\end{aligned} \tag{33}$$

Combining (32) and (33), and using the update formula for v_{ii}^{k+1} and $z_{i,-i}^{k+1}$ in Algorithm 1, we obtain

$$\begin{aligned}&\|\mathbf{z}^{k+1}-\mathbf{x}^*\|_{\pi_{k+1}}^2\\&\le\|(W_k\mathbf{z}^k-\mathbf{x}^*)-(\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\mathbf{x}^*))\|_{\pi_{k+1}}^2\\&=\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}^2+\|\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\|_{\pi_{k+1}}^2\\&\quad-2\langle W_k\mathbf{z}^k-\mathbf{x}^*,\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}},\end{aligned}$$

where the mapping $\mathbf{F}_\alpha(\cdot)$ is as defined in (23). We apply Lemma 5 to bound the second term in the preceding relation, as follows:

$$\|\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\|^2_{\pi_{k+1}} \le \mathbf{L}_\alpha^2\|W_k\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_{k+1}},$$

where $\mathbf{L}_\alpha=\sqrt{\max_{i\in[m]}\{\alpha_i^2\left(L_{-i}^2+L_i^2\right)\}}$. Therefore,

$$\begin{aligned}\|\mathbf{z}^{k+1}-\mathbf{x}^*\|^2_{\pi_{k+1}} &\le (1+\mathbf{L}_\alpha^2)\|W_k\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_{k+1}}\\ &\quad-2\langle W_k\mathbf{z}^k-\mathbf{x}^*,\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}}. \end{aligned}\tag{34}$$

To estimate the inner product in (34), we write

$$\begin{aligned}&\langle W_k\mathbf{z}^k-\mathbf{x}^*,\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}}\\ =&\langle W_k\mathbf{z}^k-\mathbf{x}^*,\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)\rangle_{\pi_{k+1}}\\ &+\langle W_k\mathbf{z}^k-\hat{\mathbf{z}}^k,\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}}\\ &+\langle \hat{\mathbf{z}}^k-\mathbf{x}^*,\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}}. \end{aligned}\tag{35}$$

Applying the Cauchy–Schwarz inequality and Lemma 5, we further obtain

$$\begin{aligned}&\left|\langle W_k\mathbf{z}^k-\mathbf{x}^*,\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)\rangle_{\pi_{k+1}}\right|\\ \le&\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}\|\mathbf{F}_\alpha(W_k\mathbf{z}^k)-\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)\|_{\pi_{k+1}}\\ \le&\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}. \end{aligned}\tag{36}$$

Similarly,

$$\begin{aligned}&\left|\langle W_k\mathbf{z}^k-\hat{\mathbf{z}}^k,\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}}\right|\\ \le&\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\|\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\|_{\pi_{k+1}}\\ \le&\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_{k+1}}. \end{aligned}\tag{37}$$

Next, we use Assumption 1(c) to estimate the last inner product in (35), as follows:

$$\begin{aligned}&\langle \hat{\mathbf{z}}^k-\mathbf{x}^*,\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}}\\ =&\sum_{i=1}^m[\pi_{k+1}]_i\alpha_i\langle[\hat{z}^k_{\pi_k}]_i-x_i^*,\nabla_iJ_i(\hat{z}^k_{\pi_k})-\nabla_iJ_i(x^*)\rangle\\ \ge&\sum_{i=1}^m[\pi_{k+1}]_i\alpha_i\mu_i\|[\hat{z}^k_{\pi_k}]_i-x_i^*\|^2\\ \ge&\min_{i\in[m]}\{[\pi_{k+1}]_i\alpha_i\mu_i\}\|\hat{z}^k_{\pi_k}-x^*\|^2. \end{aligned}$$

Note that $\|\hat{z}^k_{\pi_k}-x^*\|^2=\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|^2_{\pi_k}$, which follows by the definitions of $\hat{\mathbf{z}}^k$ and $\mathbf{x}^*$ in (31) and the fact that $\pi_k>0$ is a stochastic vector. Thus,

$$\begin{aligned}&\langle \hat{\mathbf{z}}^k-\mathbf{x}^*,\mathbf{F}_\alpha(\hat{\mathbf{z}}^k)-\mathbf{F}_\alpha(\mathbf{x}^*)\rangle_{\pi_{k+1}}\\ &\qquad\ge\min_{i\in[m]}\{[\pi_{k+1}]_i\alpha_i\mu_i\}\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|^2_{\pi_k}. \end{aligned}\tag{38}$$

Upon substituting estimates (35)–(38) back into (34), we obtain

$$\begin{aligned}\|\mathbf{z}^{k+1}-\mathbf{x}^*\|^2_{\pi_{k+1}}\le&\left(1+\mathbf{L}_\alpha^2\right)\|W_k\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_{k+1}}\\ &-2\min_{i\in[m]}\{[\pi_{k+1}]_i\alpha_i\mu_i\}\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|^2_{\pi_k}\\ &+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\\ &+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_{k+1}}. \end{aligned}$$

Under Assumption 2 and Assumption 3, by Lemma 3, the absolute probability sequence $\{\pi_k\}$ has entries uniformly bounded away from zero, implying that $\boldsymbol{\beta}_\alpha=\min_{k\ge0}\min_{i\in[m]}\{[\pi_{k+1}]_i\alpha_i\mu_i\}>0$. The stated relation follows by using $\boldsymbol{\beta}_\alpha$ in the preceding relation. ∎

Now, we provide our convergence result for Algorithm 1.

Theorem 1. *Let Assumption 1, Assumption 2, and Assumption 3 hold. Consider Algorithm 1 and the notation in* (31). *We then have for all* $k\ge0$,

$$\|\mathbf{z}^{k+1}-\mathbf{x}^*\|^2_{\pi_{k+1}}\le\lambda_{\max}(\bar{Q}_\alpha)\|\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_k},$$

where $\lambda_{\max}(\bar{Q}_\alpha)$ *is the largest eigenvalue of the matrix* $\bar{Q}_\alpha$, *which is given by*

$$\bar{Q}_\alpha=\begin{bmatrix}1-2\boldsymbol{\beta}_\alpha+\mathbf{L}_\alpha^2 & 2\sqrt{1-\boldsymbol{\eta}}\mathbf{L}_\alpha\\ 2\sqrt{1-\boldsymbol{\eta}}\mathbf{L}_\alpha & (1+2\mathbf{L}_\alpha+\mathbf{L}_\alpha^2)(1-\boldsymbol{\eta})\end{bmatrix},$$

with $\boldsymbol{\beta}_\alpha=\min_{k\ge0}\min_{i\in[m]}\{[\pi_{k+1}]_i\alpha_i\mu_i\}>0$ *and* $\mathbf{L}_\alpha=\sqrt{\max_{i\in[m]}\{\alpha_i^2\left(L_{-i}^2+L_i^2\right)\}}$. *In particular, if the stepsizes* α_i, $i\in[m]$, *are chosen such that* $\lambda_{\max}(\bar{Q}_\alpha)<1$, *then*

$$\lim_{k\to\infty}\|\mathbf{z}^k-\mathbf{x}^*\|=0,\qquad \lim_{k\to\infty}\|x^k-x^*\|=0.$$

Proof. By Lemma 7, we have that for all $k\ge0$,

$$\begin{aligned}\|\mathbf{z}^{k+1}-\mathbf{x}^*\|^2_{\pi_{k+1}}\le&\left(1+\mathbf{L}_\alpha^2\right)\|W_k\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_{k+1}}-2\boldsymbol{\beta}_\alpha\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|^2_{\pi_k}\\ &+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\\ &+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_{k+1}}. \end{aligned}$$

From the definition of $\hat{\mathbf{z}}^k$ and $\mathbf{x}^*$ in (31), it follows that

$$\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_{k+1}}=\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_k},\tag{39}$$

implying that for all $k\ge0$,

$$\begin{aligned}\|\mathbf{z}^{k+1}-\mathbf{x}^*\|^2_{\pi_{k+1}}\le&\left(1+\mathbf{L}_\alpha^2\right)\|W_k\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_{k+1}}-2\boldsymbol{\beta}_\alpha\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|^2_{\pi_k}\\ &+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\mathbf{x}^*\|_{\pi_{k+1}}\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\\ &+2\mathbf{L}_\alpha\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_{k+1}}\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_k}. \end{aligned}\tag{40}$$

The main idea of the rest of the proof is to determine the evolution relations for the quantity $\|\mathbf{z}^{k+1}-\mathbf{x}^*\|_{\pi_{k+1}}$ in terms of $\|\mathbf{z}^k-\hat{\mathbf{z}}^k\|_{\pi_k}$ and $\|\hat{\mathbf{z}}^k-\mathbf{x}^*\|_{\pi_k}$. Toward this end, we employ Lemma 6 with $W=W_k$, $z_i=z_i^k$, and stochastic vectors $\phi=\pi_{k+1}>0$, $\pi=\pi_k>0$ (which satisfy $\pi'_{k+1}W_k=\pi'_k$ by Lemma 3). Thus, using the notation in (31), we obtain for any vector $u\in\mathbb{R}^n$ and for all $k\ge0$,

$$\|W_k\mathbf{z}^k-\mathbf{1}_mu'\|^2_{\pi_{k+1}}\le\|\mathbf{z}^k-\mathbf{1}_mu'\|^2_{\pi_k}-\eta_k\|\mathbf{z}^k-\hat{\mathbf{z}}^k\|^2_{\pi_k}\tag{41}$$

(see also Remark 3). By letting $u=x^*$ and using notation $\mathbf{x}^*=\mathbf{1}_m(x^*)'$ (see (31)), we have that

$$\|W_k\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_{k+1}}\le\|\mathbf{z}^k-\mathbf{x}^*\|^2_{\pi_k}-\eta_k\|\mathbf{z}^k-\hat{\mathbf{z}}^k\|^2_{\pi_k}.\tag{42}$$

By using relation (41) with $u=\hat{z}^k$ and notation $\hat{\mathbf{z}}^k=\mathbf{1}_m(\hat{z}^k_{\pi_k})'$ (see (31)), we obtain

$$\|W_k\mathbf{z}^k-\hat{\mathbf{z}}^k\|^2_{\pi_{k+1}}\le(1-\boldsymbol{\eta})\|\mathbf{z}^k-\hat{\mathbf{z}}^k\|^2_{\pi_k},\tag{43}$$

where we also use the definition of $\boldsymbol{\eta}$ in (30). Finally, by Corollary 1, with $\gamma_i=[\pi_k]_i$, $u_i=z_i^k$, using notation $\hat{\mathbf{z}}^k=\mathbf{1}_m(\hat{z}^k_{\pi_k})'$ (see (31)), we see that for any $u\in\mathbb{R}^n$,

$$\|\hat{z}^k-u\|^2=\|\mathbf{z}^k-\mathbf{1}_mu'\|^2_{\pi_k}-\|\mathbf{z}^k-\hat{\mathbf{z}}^k\|^2_{\pi_k}.$$

By letting $u = x^*$, rearranging the terms, and noting that $\|\hat{z}^k - x^*\|^2 = \|\hat{\mathbf{z}}^k - \mathbf{x}^*\|^2_{\pi_k}$, we find that

$$\|\mathbf{z}^k - \mathbf{x}^*\|^2_{\pi_k} = \|\mathbf{z}^k - \hat{\mathbf{z}}^k\|^2_{\pi_k} + \|\hat{\mathbf{z}}^k - \mathbf{x}^*\|^2_{\pi_k}. \quad (44)$$

Plugging in the relation (44) for the first term in (42), we obtain the following

$$\|W_k\mathbf{z}^k - \mathbf{x}^*\|^2_{\pi_{k+1}} \le \|\hat{\mathbf{z}}^k - \mathbf{x}^*\|^2_{\pi_k} + (1-\boldsymbol{\eta})\|\mathbf{z}^k - \hat{\mathbf{z}}^k\|^2_{\pi_k}. \quad (45)$$

Moreover, by the triangle inequality, (39) and (43), we have

$$\begin{aligned}\|W_k\mathbf{z}^k - \mathbf{x}^*\|_{\pi_{k+1}} &\le \|W_k\mathbf{z}^k - \hat{\mathbf{z}}^k\|_{\pi_{k+1}} + \|\hat{\mathbf{z}}^k - \mathbf{x}^*\|_{\pi_{k+1}} \\ &\le \sqrt{1-\boldsymbol{\eta}}\|\mathbf{z}^k - \hat{\mathbf{z}}^k\|_{\pi_k} + \|\hat{\mathbf{z}}^k - \mathbf{x}^*\|_{\pi_k}. \quad (46)\end{aligned}$$

Now, we are ready to finish the proof of the theorem. Substituting the preceding relations (43) and (46) back into (40), it follows that

$$\begin{aligned}&\|\mathbf{z}^{k+1} - \mathbf{x}^*\|^2_{\pi_{k+1}} \\ &\le \left(1 + \mathbf{L}_\alpha^2\right)\left((1-\boldsymbol{\eta})\|\mathbf{z}^k - \hat{\mathbf{z}}^k\|^2_{\pi_k} + \|\hat{\mathbf{z}}^k - \mathbf{x}^*\|^2_{\pi_k}\right) \\ &+2\mathbf{L}_\alpha(\sqrt{1-\boldsymbol{\eta}}\ \|\mathbf{z}^k - \hat{\mathbf{z}}^k\|_{\pi_k} + \|\hat{\mathbf{z}}^k - \mathbf{x}^*\|_{\pi_k})\sqrt{1-\boldsymbol{\eta}}\ \|\mathbf{z}^k - \hat{\mathbf{z}}^k\|_{\pi_k} \\ &+2\mathbf{L}_\alpha\sqrt{1-\boldsymbol{\eta}}\|\mathbf{z}^k - \hat{\mathbf{z}}^k\|_{\pi_k}\|\hat{\mathbf{z}}^k - \mathbf{x}^*\|_{\pi_k} - 2\boldsymbol{\beta}_\alpha\|\hat{\mathbf{z}}^k - \mathbf{x}^*\|^2_{\pi_k} \\ &= \begin{bmatrix}\|\hat{\mathbf{z}}^k - \mathbf{x}^*\|_{\pi_k} \\ \|\mathbf{z}^k - \hat{\mathbf{z}}^k\|_{\pi_k}\end{bmatrix}' \bar{Q}_\alpha \begin{bmatrix}\|\hat{\mathbf{z}}^k - \mathbf{x}^*\|_{\pi_k} \\ \|\mathbf{z}^k - \hat{\mathbf{z}}^k\|_{\pi_k}\end{bmatrix},\end{aligned}$$

where

$$\bar{Q}_\alpha = \begin{bmatrix}1 - 2\boldsymbol{\beta}_\alpha + \mathbf{L}_\alpha^2 & 2\sqrt{1-\boldsymbol{\eta}}\mathbf{L}_\alpha \\ 2\sqrt{1-\boldsymbol{\eta}}\mathbf{L}_\alpha & (1 + 2\mathbf{L}_\alpha + \mathbf{L}_\alpha^2)(1-\boldsymbol{\eta})\end{bmatrix}.$$

Hence,

$$\begin{aligned}\|\mathbf{z}^{k+1} - \mathbf{x}^*\|^2_{\pi_{k+1}} &\le \lambda_{\max}(\bar{Q}_\alpha)(\|\hat{\mathbf{z}}^k - \mathbf{x}^*\|^2_{\pi_k} + \|\mathbf{z}^k - \hat{\mathbf{z}}^k\|^2_{\pi_k}) \\ &= \lambda_{\max}(\bar{Q}_\alpha)\|\mathbf{z}^k - \mathbf{x}^*\|^2_{\pi_k}.\end{aligned}$$

where $\lambda_{\max}(\bar{Q}_\alpha)$ is the largest eigenvalue of the matrix $\bar{Q}_\alpha$, and the last equality is obtained by using (44). The rest of the statement follows immediately from the preceding relation. ■

Theorem 1 shows that the iterates $\{x^k\}$ of Algorithm 1 converge to the NE x^* with a geometric rate if the stepsizes α_i are suitably chosen. Moreover, the estimates z_i^k also converge to x^* with a geometric rate for all agents $i \in [m]$. To illustrate that such stepsizes can be found, lets consider the case when $\alpha = \alpha_i$ for all $i \in [m]$. Recalling that

$$\mathbf{L}_\alpha = \sqrt{\max_{i\in[m]}\{\alpha_i^2\left(L_{-i}^2 + L_i^2\right)\}},$$

$$\boldsymbol{\beta}_\alpha = \min_{k\ge 0}\min_{i\in[m]}\{[\pi_{k+1}]_i\alpha_i\mu_i\},$$

we have

$$\mathbf{L}_\alpha = \alpha L \qquad \text{with} \quad L = \sqrt{\max_{i\in[m]}\{L_{-i}^2 + (L_i^2\}}, \quad (47)$$

$$\boldsymbol{\beta}_\alpha = \alpha\delta \qquad \text{with} \quad \delta = \min_{k\ge 0}\min_{i\in[m]}\{[\pi_{k+1}]_i\mu_i\}. \quad (48)$$

It follows that $\lambda_{\max}(\bar{Q}_\alpha) < 1$ if and only if the matrices $\bar{Q}_\alpha$ and $I - \bar{Q}_\alpha$ are positive definite. By Sylvester's criterion, the following inequalities should hold $[\bar{Q}_\alpha]_{1,1} > 0$, $\det(\bar{Q}_\alpha) > 0$, $[I - \bar{Q}_\alpha]_{1,1} > 0$, and $\det(I - \bar{Q}_\alpha) > 0$. These inequalities, respectively, yield the following conditions for α:

$$L^2\alpha^2 - 2\delta\alpha + 1 > 0 \quad (49)$$

$$\begin{aligned}&(1-\boldsymbol{\eta})[L^4\alpha^4 + 2L^2(L-\delta)\alpha^3 \\ &-2L(L+2\delta)\alpha^2 + 2(L-\delta)\alpha + 1] > 0 \quad (50)\end{aligned}$$

$$-L^2\alpha^2 + 2\delta\alpha > 0 \quad (51)$$

$$\begin{aligned}&\alpha[L^4(1-\boldsymbol{\eta})\alpha^3 + 2L^2(L-\delta)(1-\boldsymbol{\eta})\alpha^2 \\ &-(4L(L+\delta)(1-\boldsymbol{\eta}) + L^2\boldsymbol{\eta})\alpha + 2\boldsymbol{\eta}\delta] > 0 \quad (52)\end{aligned}$$

with L, δ given in (47)–(48) and $\boldsymbol{\eta}$ is defined in (30).

Note that the strong monotonicity constant μ_i and the Lipschitz constant for the gradient mapping $\nabla_i J_i(\cdot, x_i)$ always satisfy $\mu_i \le L_i$ (see Assumption 1). By the definition of L in (47), it follows that $\mu_i \le L$ for all i. Since each π_k is a stochastic vector, by the definition of δ in (48), it follows that $\delta < L$. As a consequence, the inequality (49) holds for any $\alpha \in \mathbb{R}$.

Solving (51) leads to

$$0 < \alpha < \frac{2\delta}{L^2}. \quad (53)$$

Moreover, since the constant terms of the polynomials in (50) and (52) are positive, we can choose stepsize α small enough that satisfies the two inequalities. Specifically, we have $1 - \boldsymbol{\eta} > 0$ and $L - \delta > 0$, thus, (50) holds when

$$L^4\alpha^4 - 2L(L+2\delta)\alpha^2 + 1 > 0,$$

which gives

$$\begin{aligned}&\alpha < \frac{L(L+2\delta) - 2L\sqrt{L\delta + \delta^2}}{L^4} \\ \text{or} \quad &\alpha > \frac{L(L+2\delta) + 2L\sqrt{L\delta + \delta^2}}{L^4}. \quad (54)\end{aligned}$$

The inequality in (52) holds if the quantity inside the square brackets is positive. By rearranging terms and factoring $(1-\boldsymbol{\eta})$ out, we obtain:

$$(1-\boldsymbol{\eta})\Big[L^4\alpha^2 + 2L^2(L-\delta)\alpha - 4L(L+\delta)\Big]\alpha + \boldsymbol{\eta}(2\delta - L^2\alpha) > 0$$

From (53) we have $2\delta - L^2\alpha > 0$ and since $\boldsymbol{\eta} \in (0, 1)$, we can either choose α very close to 0 so that the term $\boldsymbol{\eta}(2\delta - L^2\alpha)$ dominates or we can require the following inequality to hold

$$L^4\alpha^2 + 2L^2(L-\delta)\alpha - 4L(L+\delta) > 0,$$

which is equivalent to have

$$\alpha > \frac{-(L-\delta) + \sqrt{5L^2 + 2L\delta + \delta^2}}{L^2}, \quad (55)$$

if it satisfies (53) and (54).

Remark 4. *Theorem 1 applies to the special case when the communication network is directed and static, in which case we typically use $W_k = W$ for all $k \ge 1$. When the underlying graph is strongly connected and W is compliant with the graph structure, it is well-known that W has a unique stochastic left-eigenvector $\pi > \mathbf{0}$, associated with the simple eigenvalue 1, i.e., $\pi' W = \pi'$. In this case, the convergence result for Algorithm 1 follows immediately from Theorem 1 where $\pi_k = \pi$ for all $k \ge 0$.*

Remark 5. *Theorem 1 uses the assumption that the sequence* $\{G_k\}$ *consists of strongly connected graphs. It is possible to relax this assumption by considering* B*-strongly-connected graph sequence, where it is assumed that there is an integer* $B \geq 10$ *such that the graph with edge set*

$$E_k^B = \bigcup_{i=kB}^{(k+1)B-1} \mathsf{E}_i$$

is strongly connected for every $k \geq 0$*. In this case, the analysis can make use of Theorem 4.20 in [21] stating that there exist a set of absolute probability vectors* $\{\pi_k\}$ *such that*

$$\pi'_{k+r}\left(W_{k+r-1}\cdots W_{k+1}W_k\right) = \pi'_k.$$

Also, Lemma 3(b) can be extended to show that

$$[\pi_k]_i \geq \frac{w^{mB}}{m} \qquad \text{for all } i \in [m] \text{ and all } k \geq 0 \quad .$$

With the use of these results, the rest of convergence analysis of Algorithm 1 will follow similarly to our analysis for strongly connected graphs.

VII. Numerical results

In this section, we evaluate the performance of the proposed approach through a Nash-Cournot game, as described in [15], over several types of communication networks. Specifically, consider a set of m firms (i.e., agents) involved in the production of a homogeneous commodity. The firms compete over N markets, $M_1, \ldots, M_N$, as presented in Figure 1. Denote the market index by h, where $h \in [N] = \{1, 2, \ldots, N\}$. Firm i, $i \in [m] = \{1, 2, \ldots, m\}$ participates in the competition in n_i markets, where $n_i \leq N$ is a non-negative integer number, by deciding the amount of commodity $x_i \in \Omega_i = [\mathbf{0}, \mathbf{C}_i]$, where $\mathbf{C}_i \in \mathbb{R}^{n_i}$, to be produced and delivered. We study a distributed partial-information setting where there is no centralized communication system between firms, however, they may communicate with a local subset of neighbouring firms via some underlying communication network $\mathbb{G}_k$ that may vary over time.

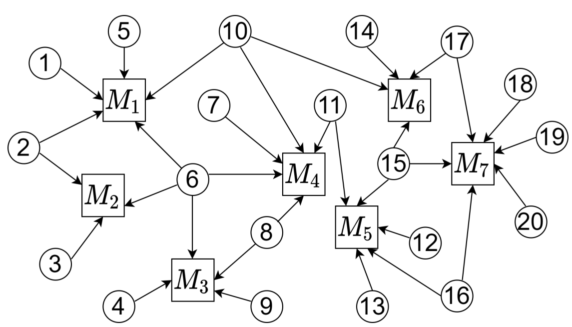


Fig. 1: Network Nash-Cournot game: An edge from i to M_h on this graph implies that firm i participates in market M_h.

Firm i has a local matrix $B_i \in \mathbb{R}^{N \times n_i}$, with elements 0 and 1 indicating which markets it participates in, i.e., we have:

$$[B_i]_{hj} = \begin{cases} 1, & \text{if agent i delivers } [x_i]_j \text{ to } M_h, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the matrices $B_1, \ldots, B_m$ can be interpreted as bipartite graphs specifying the connections between the firms and the markets. Let $n = \sum_{i=1}^m n_i$, $x = [x_i]_{i \in [m]} \in \mathbb{R}^n$, and $B = [B_1, \cdots, B_m] \in \mathbb{R}^{N \times n}$. Then, given an action profile x of all the firms, the vector of the total product supplied to the markets can be expressed as $Bx = \sum_{i=1}^m B_i x_i \in \mathbb{R}^N$.

Similar to [15], we consider a setting with $m = 20$ firms and $N = 7$ markets. Also, the commodity's price in market M_h is assumed to be a linear function of the total amount of commodity supplied to the market [15], i.e., we have:

$$p_h(x) = \bar{P}_h - \chi_h [Bx]_h, \quad \forall h,$$

where $\bar{P}_h > 0$ and $\chi_h > 0$. This function implies that the price decreases as the amount of supplied commodity increases. Let $\bar{P} = [\bar{P}_h]_{h=\overline{1,N}} \in \mathbb{R}^N$ and $\Xi = \mathrm{diag}([\chi_{\mathrm{h}}]_{\mathrm{h}=\overline{1,\mathrm{N}}}) \in \mathbb{R}^{\mathrm{N}\times\mathrm{N}}$. Then, the price vector function $P = [p_h]_{h=\overline{1,N}}$ that maps the total supply of each market to the corresponding price, i.e. $P : \mathbb{R}^N \mapsto \mathbb{R}^N$, has the following form:

$$P = \bar{P} - \Xi A x,$$

and $P' B_i x_i$ is the payoff of firm i obtained by selling x_i to the markets that it connects with. Firm i's production cost $c_i(\cdot) : \Omega_i \mapsto \mathbb{R}$ is a strongly convex, quadratic function:

$$c_i(x_i) = x_i' Q_i x_i + q_i' x_i,$$

with $Q_i \in \mathbb{R}^{n_i \times n_i}$ symmetric and $Q_i \succ 0$, and $q_i \in \mathbb{R}^{n_i}$. Thus, the local objective function of firm i, which depends on the other firms' production profile x_{-i}, can be given as:

$$J_i(x_i, x_{-i}) = c_i(x_i) - (\bar{P} - \Xi A x)' B_i x_i, \quad \forall i, \tag{56}$$

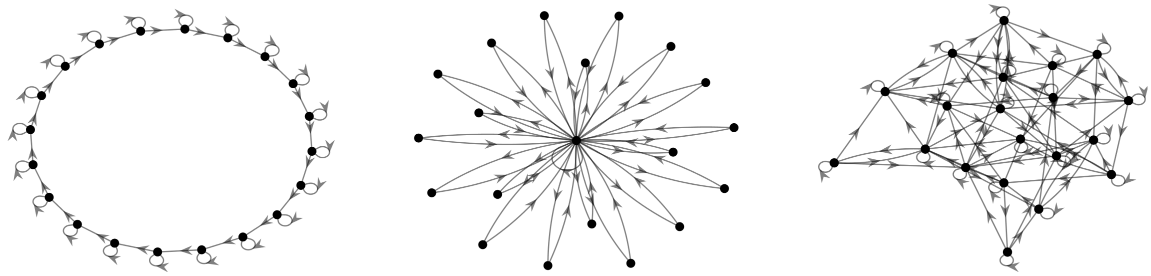
so that for all i,

$$\nabla_i J_i(x) = 2Q_i x_i + q_i + B_i' \Xi B_i x_i - B_i'(\bar{P} - \Xi B x). \tag{57}$$

For the numerical experiments, we consider $\mathbf{0} \leq x_i \leq \mathbf{C_i}$, where each component of $\mathbf{C_i}$ is generated uniformly at random from the interval $[5, 10]$. In the production cost function $c_i(x_i)$, Q_i is diagonal with its entries uniformly distributed in $[1, 8]$ and each component q_i is randomly drawn from a uniform distribution on $[1, 2]$. In the price function $P(Bx)$, $\bar{P}_h$ and χ_h are chosen uniformly at random from the intervals $[10, 20]$ and $[1, 3]$, respectively.

We consider a partial-information scenario in which firms can communicate with a local subset of neighbors via a directed communication network $\mathbb{G}_k$ at time k. Figure 2 depicts the communication topologies used in our experiments, including a directed ring, an undirected star, and a randomly generated graph that underline the communication network among 20 firms. We first consider static communication networks. Then, the performance comparison between the static case and the time-varying case is examined. Finally, we investigate time-varying directed communication networks and compare the performance of our proposed algorithm with the state-of-the-art algorithm (i.e., Algorithm 1 in [5]).

All the graphs used in our simulations are strongly connected. For each random graph, we create a directed connection (a ring or a cycle) going through all agents to ensure the strongly connected property. It is worth emphasizing that the communication network is different from the Nash-Cournot game network. While the former specifies how the agents exchange their information, the latter represents the Nash-Cournot game which can be translated into specific utility



(a) Ring (b) Star (c) Random

Fig. 2: Communication network topologies

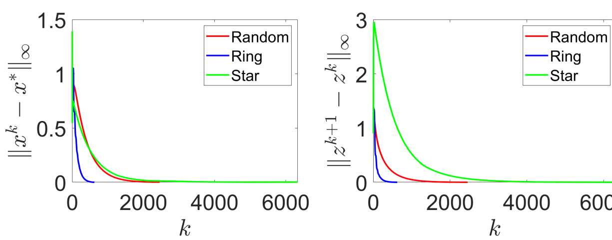
functions and constraints in the optimization problem governing the game. The adjacency matrix $[A_k]_{ij}$ corresponding to each graph $\mathbb{G}_k$, which indicates whether agent j can communicate with agent i, is defined as follows:

$$\begin{cases} [A_k]_{ij} = 1, & \text{when} \quad j \in \mathcal{N}_{ik}^{\text{in}}, \\ [A_k]_{ij} = 0, & \text{otherwise}. \end{cases}$$

We also define the row-stochastic weight matrix W_k according to each A_k such that:

$$[W_k]_{ij} = \begin{cases} 0, & \text{if} \quad [A_k]_{ij} = 0, \\ \delta, & \text{if} \quad [A_k]_{ij} = 1 \quad \text{and} \quad i \neq j, \\ 1 - \delta d_k(i), & \text{if} \quad i = j, \end{cases}$$

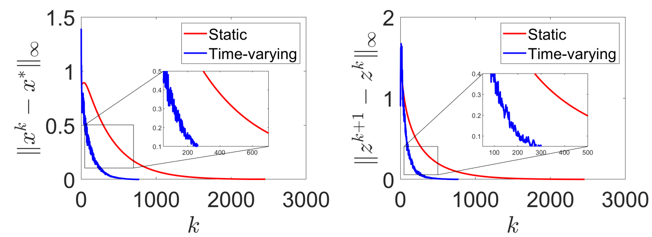
where $d_k(i)$ represents the number of agents communicating with agent i at iteration k, i.e., $d_k(i) = |\mathcal{N}_{ik}^{\text{in}}|$. Additionally, let $\delta = \dfrac{0.5}{\max_{i,k}\{d_k(i)\}}$. By using the adjacency matrix A_k and the weight matrix W_k, each agent can update its estimates of the other agents' decisions and its own decisions.



(a) $\|x^k - x^*\|_\infty$ (b) $\|\mathbf{z}^{k+1} - \mathbf{z}^k\|_\infty$

Fig. 3: Plots of errors when using different communication graphs (a random graph, a directed ring, and an undirected star). The numerical results are computed using Algorithm 1 with $\alpha = 0.05$. Left: The errors between the actions generated by the algorithm and the NE. Right: The errors between the two estimation matrices $\mathbf{z}$ of all agents' actions by other agents including themselves at two consecutive iterations, which is also known as the consensus error.

We will use our proposed algorithm (i.e., Algorithm 1) and [5, Alg. 1] (i.e., Algorithm 1 in [5]) to solve the Nash-Cournot game above. The algorithms terminate if $\|x^{k+1} - x^k\|_\infty = \max_i |[x^{k+1}]_i - [x^k]_i|$ and $\|\mathbf{z}_{k+1} - \mathbf{z}_k\|_\infty = \max_{i,j} |[\mathbf{z}_{k+1}]_{ij} - [\mathbf{z}_k]_{ij}|$ are sufficiently small (i.e., less than 10^{-3} in our experiments) or the number of iterations exceeds 100000 iterations.



(a) $\|x^k - x^*\|_\infty$ (b) $\|\mathbf{z}^{k+1} - \mathbf{z}^k\|_\infty$

Fig. 4: Plots of errors for a randomly generated graph when the network is static versus when it varies over time. The results are computed using Algorithm 1 with stepsize $\alpha = 0.05$.

Note that, in [5], the authors proposed 2 algorithms. However, [5, Alg. 2] uses the estimates of the PF eigenvector q. Hence, it is sufficient for us to compare our proposed algorithm with [5, Alg. 1] only.

First, we consider the static case, where the communication graph is time-invariant, and compare the performance of the proposed algorithm over different types of connectivity (Figure 2). The impact of the time-varying nature of the connectivity graph is also explored by considering time-varying directed graphs. The communication topologies used in this test include the following:

- A randomly generated graph where each firm can send its information to 4 other firms,
- A directed ring or cycle,
- An undirected star with firm 1 as the center,
- A sequence of time-varying directed graphs.

All considered graphs are strongly connected, with self-loops at every node. The stepsize α in Algorithm 1 is set to be 0.05. Figure 3 illustrates the convergence properties of the proposed algorithm over different connectivity graphs. It can be observed that the convergence rate is fastest for the ring connectivity and slowest for the star-shaped network. This is as expected since for the star-shaped network, all the information needs to go through firm 1 before reaching other agents. Thus, this process increases the number of communications needed to exchange information among the firms. Further details of the convergence are shown in Table I.

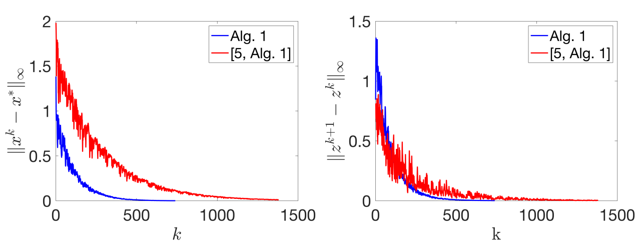
Figure 4 compares the convergence of the proposed algorithm when the communication network is static versus when it is time-varying. An interesting observation is that when a sequence of time-varying directed graphs is used, it converges

Graph Type	# Iterations	Running Time (s)	$\|x^k - x^*\|_\infty$	$\|\mathbf{z}^{k+1} - \mathbf{z}^k\|_\infty$
Static Random	2455	0.258	4.2e-6	9.9e-4
Static Ring	622	0.070	2.1e-5	9.9e-4
Static Star	6332	0.622	4.1e-7	9.9e-4
Time-varying Random	774	0.121	2.2e-4	9.7e-4

TABLE I: Performance of Algorithm 1 when using different types of communication graphs with stepsize $\alpha = 0.05$. This table displays the total number of iterations and the running time for each algorithm as well as their performance in terms of the errors at the last iteration, for one game instance.

faster. Intuitively, by varying the connections among all the firms, the firms have more opportunities to exchange and update information with more firms. Thus, from the results, the proposed algorithm performs better with time-varying graphs compared to a static graph.

Next, we compare the performance of the proposed algorithm with the state-of-the-art algorithm in [5, Alg. 1] for time-varying directed communication graphs. At every iteration, we randomly generate a strongly connected directed graph with self-loops. We assume that each firm can communicate with at most 4 neighboring firms to keep the communication network from being too dense. We run the two algorithms with 3 different choices of stepsize $\alpha = 0.1$, $\alpha = 0.05$, and $\alpha = 0.01$, and conduct 1000 simulations for each choice of α. Note that the communication graph changes at every iteration. Additionally, in our proposed algorithm, all firms use the same constant stepsize α. On the other hand, [5, Alg. 1] uses different stepsizes for different firms where $\alpha_i = \dfrac{\alpha * \min \nu_i}{\nu_i}$, $\forall i$, with ν_i is an element of the Perron-Frobenius (PF) eigenvector of the adjacency matrix. We scale all the stepsizes that are used in [5, Alg. 1] so that they are not too big to ensure convergence. Indeed, [5, Alg. 1] requires the knowledge of the communication network and the computation of the PF eigenvector at every iteration, which is impractical in many situations and computationally more expensive than our algorithm, as will be shown in the simulation results.



(a) $\|x^k - x^*\|_\infty$ (b) $\|\mathbf{z}^{k+1} - \mathbf{z}^k\|_\infty$

Fig. 5: Convergence property for one game instance. The numerical results are computed using Algorithm 1 and [5, Alg. 1] with $\alpha = 0.05$. Left: gaps between the actions generated by the algorithms and the NE. Right: Consensus errors.

Figure 5 illustrates the convergence of Algorithm 1 and [5, Alg. 1] for a certain game instance (i.e., simulation number 1 in Table II). The actions x^k and the estimates $\mathbf{z}^k$ in the plot are computed with $\alpha = 0.05$. The stopping criteria is such that the two errors $\|x^k - x^*\|_\infty$ and $\|\mathbf{z}^{k+1} - \mathbf{z}^k\|_\infty$ are less than 0.001. Algorithm 1 takes 0.243 seconds and stops after 740 iterations. For the verification purpose, we first find the NE x^* under the situation where there is a centralized communication system that broadcasts the information of all agents. Thus, all agents have access to the decisions and estimations of all other agents. In this case, we assume that all information are equally taken into account by setting the weight matrix $W = \dfrac{1}{N}\mathbf{1}\mathbf{1}'$. It is shown that under full information, the sequence of decision updates converges to the NE. Also, the obtained actions satisfy the condition $\langle \nabla J_i(x^*), x_i - x_i^* \rangle \geq 0,\ \forall x_i \in \Omega_i,\ \forall i \in [m]$, which ensures the obtained actions correspond to the NE.

The left plot of Figure 5 shows the gaps between the action x^k generated by the two algorithms and the NE x^* at each iteration k. The right plot shows the errors between the two estimation matrices at two consecutive iterations, i.e., $\mathbf{z}^k$ and $\mathbf{z}^{k+1}$, which are also known as the consensus errors. The errors are computed using the max norm. We can see that the sequence of actions x^k generated by Algorithm 1 converges to the NE x^* as expected and the consensus error is also decreasing to 0. The behavior of the consensus error implies that all agents' estimates of the other agents' actions as well as their own actions, reach consensus after a while. Figure 5 also shows the convergence behavior of [5, Alg. 1]. Algorithm 1 performs comparably with with [5, Alg. 1] without using the knowledge of the communication matrix and the computation of the PF eigenvector of the adjacency matrices. Indeed, Algorithm 1 is more computationally efficient than [5, Alg. 1] which takes 1380 iterations and takes 1.631 seconds.

Table II further compares the the performance of Algorithm 1 to [5, Alg. 1] over 5 random game instances. We observe that Algorithm 1 outperforms [5, Alg. 1] in terms of the total running time. This is expected as in [5, Alg. 1], the value of the PF eigenvector is required at each iteration, which is computationally expensive, especially for larger systems and communication networks. Meanwhile, in our proposed algorithm, we do not require the knowledge of the PF eigenvector. We also run 1000 simulations for each choice of $\alpha = 0.01, 0.05$, and 0.1. The results are reported in Table III by calculating the mean of the results obtained from running 1000 simulations. On the average, as presented in Table III, our algorithm requires less iterations and takes less time to converge compared to those of [5, Alg. 1].

VIII. Conclusions and Future Work

In this paper, we proposed a distributed algorithm for NE seeking under a partial information scenario by combining the idea of the distributed gradient method and decision estimation alignment using consensus. We studied the convergence of this algorithm over time-varying directed communication networks. The algorithm involves every agent performing a gradient step to minimize its own cost function while sharing and retrieving information locally among its neighbors in the network. While the existing methods assume balancedness and/or global knowledge of the network communication structure such as the Perron-Frobenius eigenvector of the adjacency matrix, our algorithm only requires row-stochasticity of the mixing matrices which is easily implementable. The analysis guarantees the convergence to the NE even when the network varies over time. Through a Nash-Cournot game, we

Simulation no.	# Iterations		Running Time (s)		$\|x^k - x^*\|_\infty$		$\|\mathbf{z}^{k+1} - \mathbf{z}^k\|_\infty$	
	Alg. 1	**[5]**	**Alg. 1**	**[5]**	**Alg. 1**	**[5]**	**Alg. 1**	**[5]**
1	740	1380	0.243	1.631	2.5e-04	1.3e-02	9.7e-04	7.4e-04
2	767	1630	0.247	1.864	2.6e-04	9.4e-03	9.8e-04	9.7e-04
3	653	1181	0.216	1.389	3.0e-04	1.1e-02	9.8e-04	8.2e-04
4	697	1302	0.226	1.517	1.5e-04	1.4e-02	9.5e-04	9.7e-04
5	757	1568	0.244	1.811	3.5e-04	1.3e-02	9.6e-04	8.7e-04

TABLE II: Performance comparison between Algorithm 1 (i.e., **Alg. 1**) and [5, Alg. 1] using $\alpha = 0.05$. The parameters defining each game are randomly generated for each simulation. This table displays the total number of iterations and the running time of each algorithm as well as their performances in terms of the errors at the last iteration, for five instances of the game.

Stepsize	Avg. # Iterations		Avg. Running Time (s)		Avg. errors $\|x^k - x^*\|_\infty$		Avg. errors $\|\mathbf{z}^{k+1} - \mathbf{z}^k\|_\infty$	
	Alg. 1	**[5]**	**Alg. 1**	**[5]**	**Alg. 1**	**[5]**	**Alg. 1**	**[5]**
$\alpha = 0.01$	1984.95	3846.31	0.6049	4.2346	0.0088	0.0649	0.0010	0.0009
$\alpha = 0.05$	673.47	1319.22	0.2328	1.6313	0.0003	0.0099	0.0009	0.0009
$\alpha = 0.1$	467.09	835.66	0.1582	0.9965	0.0011	0.0038	0.0010	0.0009

TABLE III: Performance comparison between Algorithm 1 (i.e., **Alg. 1**) and [5, Alg. 1]. For each stepsize, we conduct 1000 simulations. This table shows the average performances of the two algorithms in terms of the total number of iterations, the running time, and the errors at the last iteration.

demonstrated that our proposed algorithm is able to converge to the NE, while being more general and more computationally efficient than the state-of-the-art methods. An interesting open question is to explore the convergence while relaxing some assumptions such as the strong convexity of the cost functions. Additionally, extensions to more complex generalized game models containing local constraints and coupling constraints among agents' decisions would be of considerable interest.

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