A stable and accurate algorithm for a generalized Kirchhoff-Love plate model

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Introduction

Kirchhoff-Love model [1]:

- Developed in 1888 by Love using assumptions proposed by Kirchhoff
- One of the most common dimensionally-reduced models of a thin linearly elastic plate
- Analytical solutions are available only to a limited number of cases with simple specifications [2]

Chladni's patterns [3]:

- Show the nodal lines, where no vertical displacements occurred, of the different natural modes of vibration
- Natural mode: a pattern of motion in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation
- Plate resonates at the natural frequencies

Formulation

Kirchhoff-Love theory's assumptions

- The plate is thin
- The displacements and rotations are small
- Transverse shear strains are neglected
- The transverse normal stress is negligible compared to the other stress components

Governing Equations: $\rho h \ddot{w}(t,x,y) = -\mathcal{K}w(t,x,y) - \mathcal{B}\dot{w}(t,x,y) + f(t,x,y)$

Operators	Description
$\mathcal{K} = K_0 I - T\nabla^2 + D\nabla^4$	Time-invariant, symmetric differential operator
$\mathcal{B} = K_1 I - T_1\nabla^2$	Damping operator
$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$	Laplacian operator
$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$	Biharmonic operator

Parameters	Description
h	Constant thickness
ρ	Density
K_0	Linear stiffness coefficient
T	Tension coefficient
D	Bending stiffness
ν	Poisson's ratio
K_1	Linear damping coefficient
T_1	Visco-elastic damping coefficient

Boundary & Initial Conditions

Boundary Conditions

We consider the following three common types of physical boundary conditions for a plate:

- Clamped: $w(t,x,y) = 0, \quad \frac{\partial w}{\partial n}(t,x,y) = 0$
- Simply Supported: $w(t,x,y) = 0, \quad -D\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right)(t,x,y) = 0$
- Free: $-D\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right)(t,x,y) = 0, \quad -D\frac{\partial}{\partial n}\left[\frac{\partial^2 w}{\partial n^2} + (2-\nu)\frac{\partial^2 w}{\partial t^2}\right](t,x,y) = 0$

Initial Conditions: $w(0,x,y) = \alpha(x,y), \quad \dot{w}(0,x,y) = \beta(x,y)$, for given functions $\alpha(x,y)$ and $\beta(x,y)$

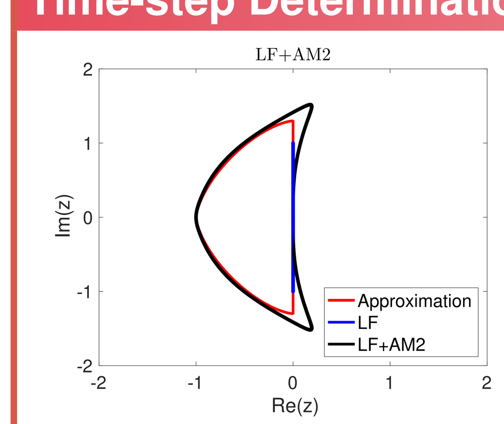
Numerical Method

Centered finite difference approximation for spatial discretization

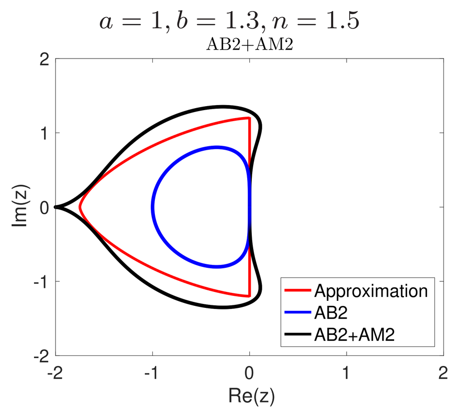
For time integration

- Explicit predictor-corrector time-stepping method:
  Predictor: Leapfrog (LF) or Adams-Bashforth (AB2); Corrector: Adams-Moulton (AM2)
- Implicit Newmark-Beta (NB) method: for $\beta = 1/4$ and $\gamma = 1/2$, the NB method is second order accurate and unconditionally stable

Time-step Determination



$a = 1, b = 1.3, n = 1.5$



$a = 1.75, b = 1.2, n = 1.5$

Region of absolute stability

Half super-ellipse to approximate the region of absolute stability:

$$\left|\frac{\Re(z)}{a}\right|^n + \left|\frac{\Im(z)}{b}\right|^n \le 1, \quad \Re(z) \le 0,$$

where $\Re(z)$, $\Im(z)$ are real and imaginary parts of z, respectively.

Stable time step:

$$\Delta t = \frac{C_{\text{cfl}}}{\left(\left|\frac{\Re(\hat{\lambda}_M)}{a}\right|^n + \left|\frac{\Im(\hat{\lambda}_M)}{b}\right|^n\right)^{1/n}}$$

where $C_{\text{cfl}} \le 1$ is the Courant-Friedrichs-Lewy (CFL) number and

$$\hat{\lambda}_M = \begin{cases} -\frac{\hat{\mathcal{B}}_M}{2} \pm i\sqrt{\hat{\mathcal{K}}_M - \left(\frac{\hat{\mathcal{B}}_M}{2}\right)^2}, & \text{if } \left(\frac{\hat{\mathcal{B}}_\omega}{2}\right)^2 - \hat{\mathcal{K}}_\omega < 0 \\ -\hat{\mathcal{B}}_M, & \text{if } \left(\frac{\hat{\mathcal{B}}_\omega}{2}\right)^2 - \hat{\mathcal{K}}_\omega > 0 \end{cases}$$

For NB: C_{cfl} can be taken as big as 100

Results I: Method of Manufactured Solution

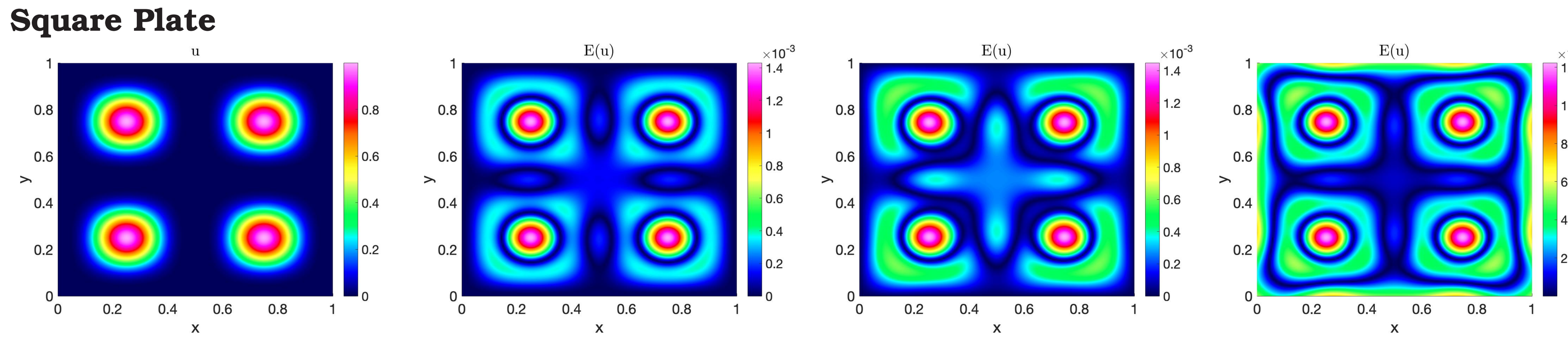
Manufactured solution: $w_e(t,x,y) = \sin^4(2\pi x)\sin^4(2\pi y)\cos(2\pi t)$

Material properties: $\rho h = 1, K_0 = 2, T = 1, D = 0.01, K_1 = 5, T_1 = 0.1, \nu = 0.1$

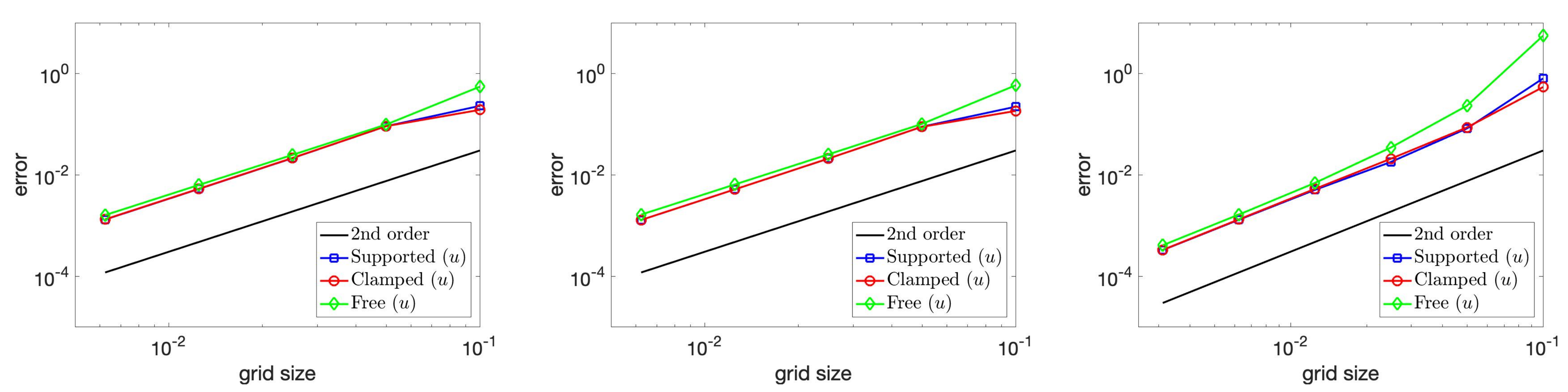
Forcing term: $f(t,x,y) = \rho h\frac{\partial^2 w_e}{\partial t^2} + K_0 w_e - T\Delta w_e + D\Delta^2 w_e + K_1\frac{\partial w_e}{\partial t} - T_1\Delta\frac{\partial w_e}{\partial t}$

Initial conditions: $w(0,x,y) = 0, \quad \dot{w}(0,x,y) = 0$

Square Plate

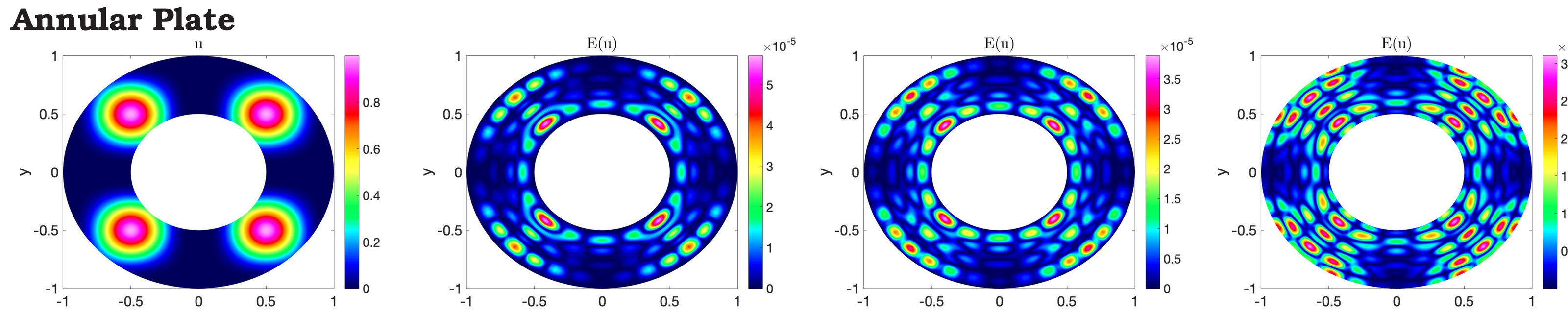


Numerical solution | Error (Clamped BC) | Error (Simply supported BC) | Error (Free BC)

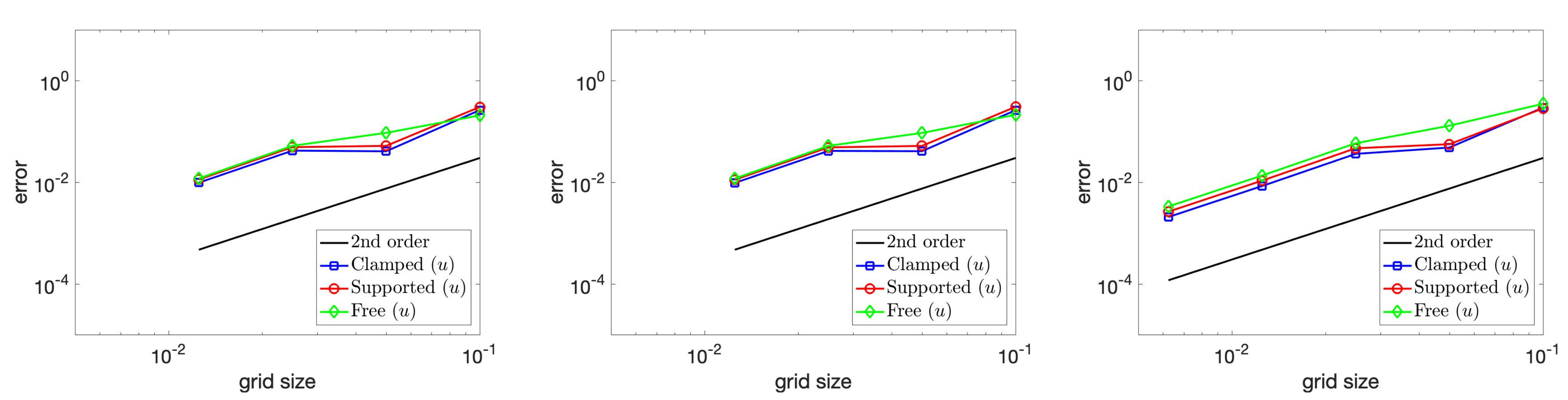


LF+AM2 | AB2+AM2 | NB

Annular Plate



Numerical solution | Error (Clamped BC) | Error (Simply supported BC) | Error (Free BC)



LF+AM2 | AB2+AM2 | NB

Results II: Standing Waves and Nodal Lines

- **Free vibration:** $f(t,x,y) = 0, \quad w(0,x,y) = \phi(x,y), \quad \dot{w}(0,x,y) = 0$
- **Force vibration:** $f(t,x,y) = F_0\cos(\Omega t)\delta_{(x-x_0, y-y_0)}, \quad w(0,x,y) = 0, \quad \dot{w}(0,x,y) = 0$

(x_0, y_0) : center of the plate

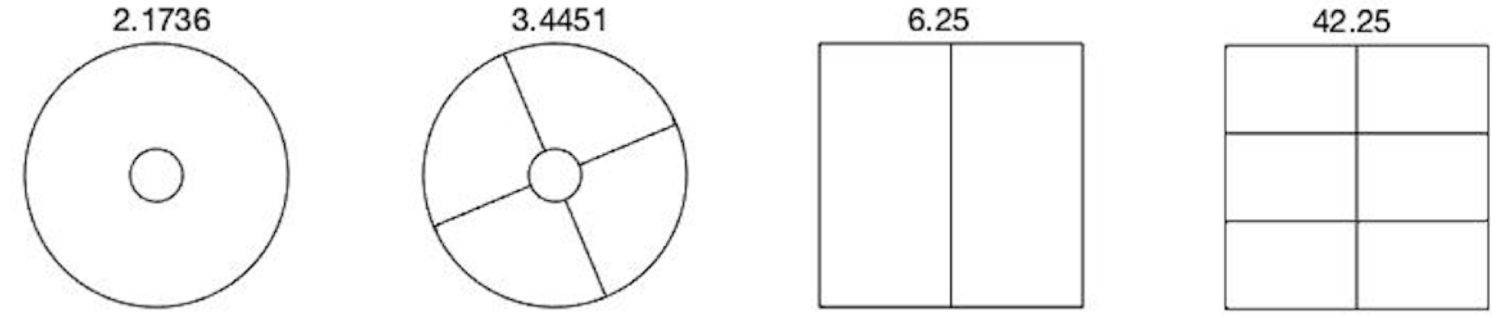
Ω and $\phi(x,y)$: eigenvalue and eigenvector of the eigenvalue problem $(K_0 I - T\nabla^2 + D\nabla^4)\phi(x,y) = \lambda\phi(x,y)$

1. Simply Supported BC

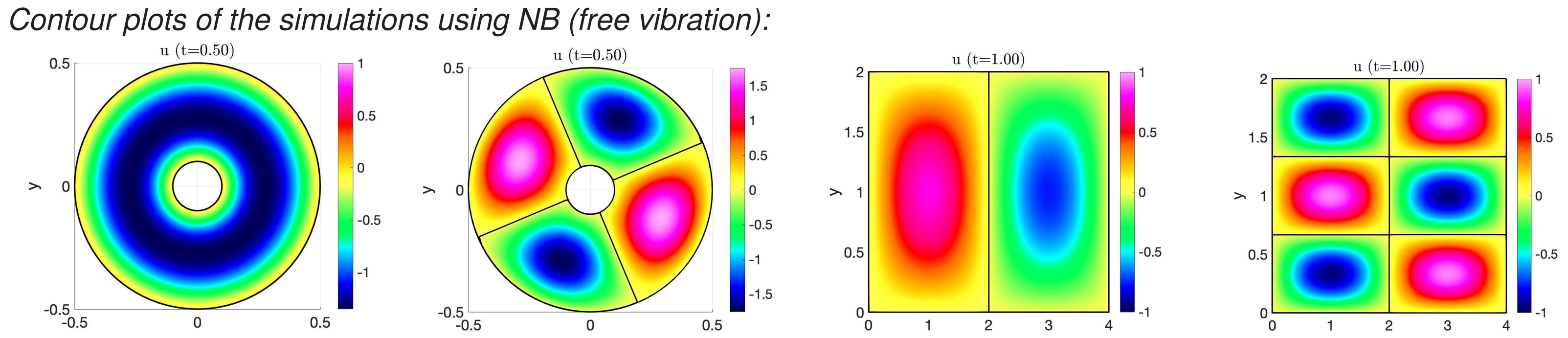
Material properties: $\rho h = 1, K_0 = 0, T = 0, D = 2, K_1 = 0, T_1 = 0, \nu = 0.1$

(Rectangular plate) Eigenvector: $\phi(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, eigenvalue: $f_{mn} = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\sqrt{\frac{D\pi^2}{4\rho h}}, m, n = 1, 2, \ldots$

Nodal line plots from the solution of the eigenvalue problem using the `eigs` *function in MATLAB:*



Contour plots of the simulations using NB (free vibration):

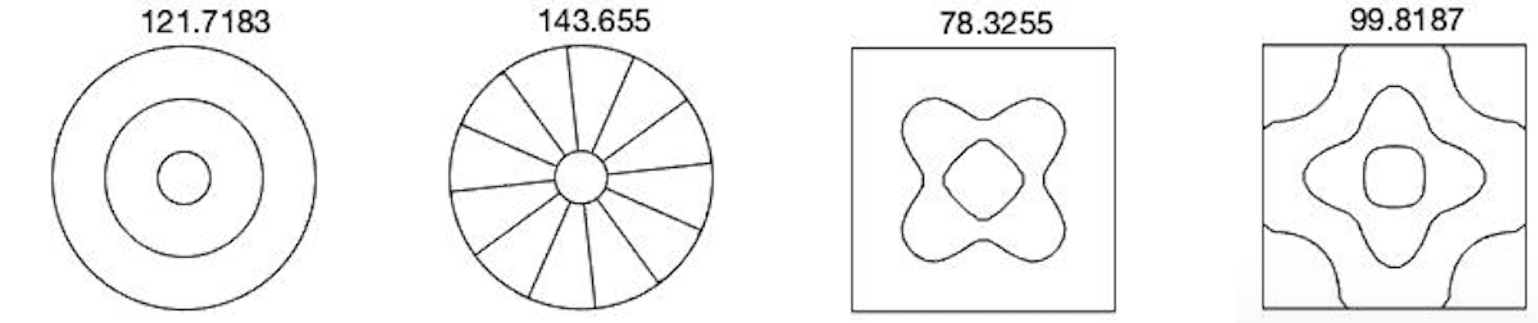


Frequency 2.1736 | Frequency 3.4451 | $m = 2, n = 1$ | $m = 3, n = 2$

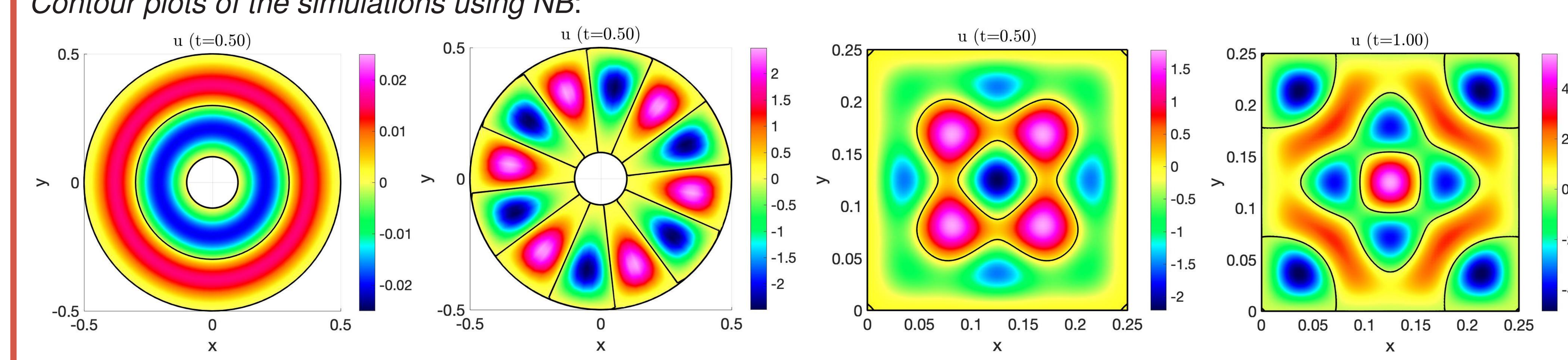
2. Clamped BC:

Material properties: $\rho h = 1, K_0 = 2, T = 1, D = 0.01, K_1 = 0, T_1 = 0, \nu = 0.1, F_0 = 10^8$

Nodal line plots from the solution of the eigenvalue problem using the `eigs` *function in MATLAB:*



Contour plots of the simulations using NB:



Frequency 121.7183 | Frequency 143.655 | Frequency 79.0282 | Frequency 100.5374

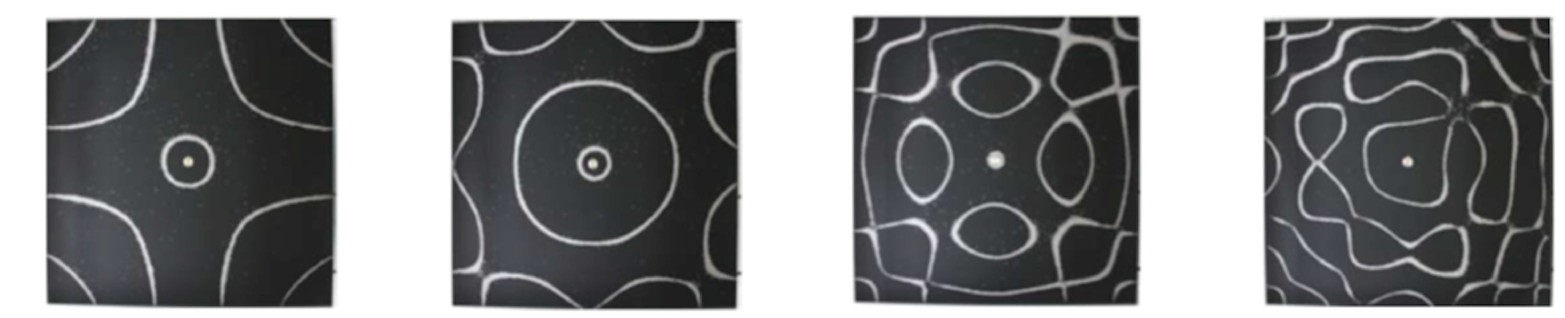
Free vibration | Forced vibration

3. Free BC:

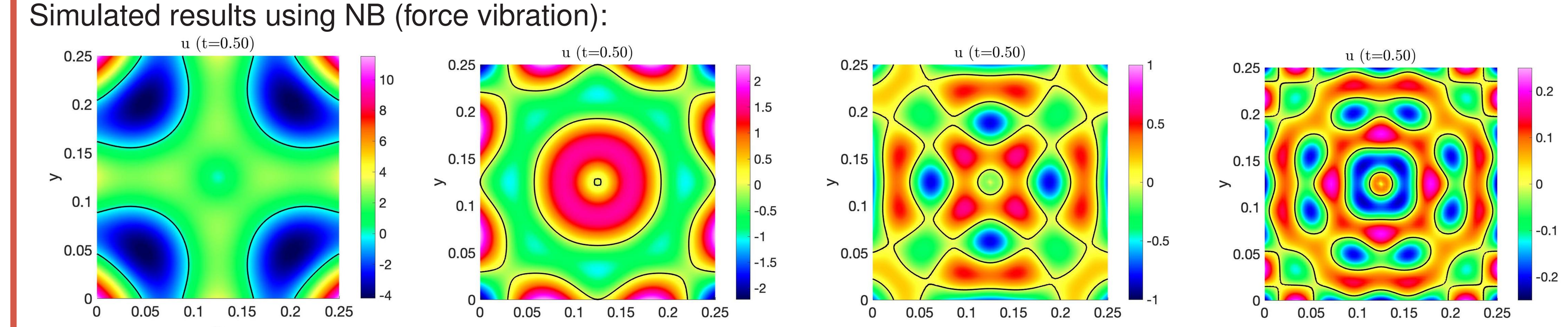
Material properties (Aluminum): $\rho = 2700, h = 0.001, K_0 = 0, T = 0, E = 69, K_1 = 0.1, T_1 = 5, \nu = 0.33, F_0 = 10^8$

The center of the plate is clamped.

Experimental results [4]:



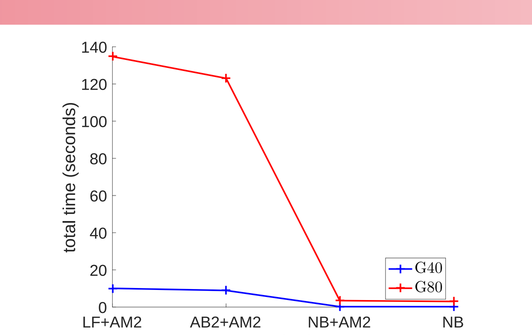
Simulated results using NB (force vibration):



Frequency $f_1 = 560.5412$ | Frequency $f_2 = 912.9099$ | Frequency $f_6 = 2037.2532$ | Frequency $f_{10} = 3159.832$

Conclusion

- A sequence of benchmark problems with increasing complexity are considered to demonstrate the numerical properties of the algorithm
- Mesh refinement study, with the method of manufactured solutions, verifies the stability, accuracy and second order convergence
- NB is more time-efficient than the explicit predictor-corrector schemes
- Nodal lines, natural mode shapes and frequencies obtained through free and forced vibrations match the expected and experimental results [4]



Total running time

Future Directions

- Extend current research to more complicated geometries
- Couple the developed plate solver with an existing fluid solver to simulate more interesting fluid-structure interaction (FSI) problems, such as blood flow in an artery

References

[1] A. E. H. Love, *On the small free vibrations and deformations of elastic shells*, Philosophical trans. of the Royal Society (London), 1888, Vol. serie A, No 17 p. 491 – 549.

[2] Rudolph Szilard, *Theories and Applications of Plate Analysis*, Classical Numerical and Engineering Methods, 2004, Wiley, p. 62 – 127.

[3] W. M. Pierce, *Chladni Plate Figures*, American Journal of Physics 19(7), 1951, DOI: 10.1119/1.1933030.

[4] P. H. Tuan, C. P. Wen, P. Y. Chiang, Y. T. Yu,H. C. Liang,K. F. Huang, Y. F. ChenKML, *Exploring the resonant vibration of thin plates: Reconstruction of Chladni patterns and determination of resonant wave numbers*, The Journal of the Acoustical Society of America 137, 2113 (2015), doi: 10.1121/1.4916704